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Wave-packet formalism of full counting statistics

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We make use of the first-quantized wave-packet formulation of the full counting statistics to describe charge transport of noninteracting electrons in a mesoscopic device. We derive various expressions for the characteristic function generating the full counting statistics, accounting for both energy and time dependence in the scattering process, and including exchange effects due to finite overlap of the incoming wave packets. We apply our results to describe the generic statistical properties of a two-fermion scattering event and find, among other features, sub-binomial statistics for nonentangled incoming states (Slater rank 1), while entangled states (Slater rank 2) may generate superbinomial (and even super-Poissonian) noise, a feature that can be used as a spin singlet-triplet detector. Another application is concerned with the constant-voltage case, where we generalize the original result of Levitov-Lesovik to account for energy-dependent scattering and finite measurement time, including short-time measurements, where Pauli blocking becomes important.

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I. INTRODUCTION

Charge transport across an obstacle in a wire is a statistical process, whose complete description is provided by the probability function $P(n,t)$, telling how many charge carriers n are transmitted through the wire during the time t. The calculation of this full counting statistics usually aims at the generating function $\chi(\lambda,t)=\Sigma_n P(n,t)e^{i\lambda n}$ for this process, from which the probability distribution $P(n,t)$ follows through simple Fourier transformation $\mathcal{F}[\chi(\lambda,t)]=P(n,t)$. The proper physical definition of the generating function $\chi(\lambda,t)$ is a nontrivial problem and has been solved by Levitov and Lesovik back in 1993,[1] see also Ref. 2, with numerous applications to follow.[3] The original definition includes a "charge counter" in the form of a spin, coupled via the gauge potential to the moving charges, and has been cast in a second-quantized formalism of appreciable complexity. The recent observation[4] of the correspondence between the generating function $\chi_1(\lambda)$ of the full counting statistics for one particle and the notion of fidelity in a (one-particle, chaotic) quantum system[5] has lead to a much simpler first-quantized formulation of full counting statistics, including the generalization $\chi_N(\lambda)$ to N particles. In fact, a first-quantized version of charge transport to calculate noise has been already introduced some years ago.[6] Furthermore, such a wave-packet formalism naturally describes the statistics of pulsed transport, where unit-flux voltage pulses generate single-particle excitations feeding the device of interest[2,7–10] (a source injecting individual electrons into a quantum wire has been realized in a recent experiment[11]). The simplicity of the first-quantized formalism then has allowed us to obtain nontrivial results on the full counting statistics for an energy-dependent scatterer, including its dependence on the exchange symmetry of the transported charge.[12]

In this paper, we make intense use of this wave-packet formalism of charge transport and (re)derive various expressions for the characteristic function $\chi_N(\lambda)$ in a much simplified manner. We start with an N-particle Slater determinant made from orthonormalized single-particle wave functions ϕ_m describing fermions incident from the left and derive the associated characteristic function describing the full counting statistics in determinant form,

$$\chi_N(\lambda)=\det\langle\phi_m|1-\mathcal{T}+\mathcal{T}e^{i\lambda}|\phi_n\rangle, \tag{1}$$

with the operator $\mathcal{T}$ describing the energy-dependent transmission across the scatterer, $\mathcal{T}=\int(dk/2\pi)T_k|k\rangle\langle k|$ in momentum (k) representation (here, the particle number N replaces the time variable t in the original formula[1]). The determinant in Eq. (1) can be cast in a product form,

$$\chi_N(\lambda)=\prod_{m=1}^{N}(1-\tau_m+\tau_m e^{i\lambda}), \tag{2}$$

where τ_m are the eigenvalues of the Hermitian operator $\mathcal{T}$ in the space spanned by the basis states $|\phi_n\rangle$. We denote the distribution in Eq. (2) as *generalized binomial*.

In a real experiment, the unit-flux voltage pulses generating the incoming wave packets may overlap. For this situation, we rederive the simple and elegant expression (2) for the full counting statistics, but with the coefficients τ_m now replaced by the roots of a generalized eigenvalue problem incorporating all effects of fermionic statistics and the full energy dependence of the transmission. Results (1) and (2) apply to a nonentangled incident state in the form of a Slater determinant;[13] an extension to include entangled states of Slater rank 2 is provided as well.[14]

Next, we generalize result (1) to describe a setup where both the scattering process and the counting window depend on time and find a compact result in form (1) with

$$\mathcal{T}\rightarrow\mathcal{T}_{\mathcal{Q}}=\mathcal{U}^{\dagger}\mathcal{Q}\mathcal{U}, \tag{3}$$

where $\mathcal{U}$ denotes the single-particle time evolution operator and the operator $\mathcal{Q}$ projects the wave function onto its measured (counted) part. Full counting statistics for fermionic atoms in determinant form has been derived in Ref. 15 through transcription of the bosonic expression[16] to the fermionic case (see Ref. 17 for a recent application).

Finally, we extend the result [Eq. (3)] to the situation where the incoming state consists of an incoherent superposition of many Slater determinants with different particle numbers. For the case of particles incident only from the left side, we find result (1) with

$$\mathcal{T} \to \eta \mathcal{T}_Q, \tag{4}$$

where η denotes the one-particle occupation-number operator. In addition, the determinant in Eq. (1) has to be taken over all the single-particle Hilbert space.

We make extended use of these formulas. For a two-particle problem, we show the following: (i) an incoming state described via a simple Slater determinant cannot generate a Fano factor $F=\langle\langle n^2\rangle\rangle/\langle n\rangle > 1-\langle n\rangle/2$ (i.e., noise is always sub-binomial and in particular also sub-Poissonian; there is no bunching); the above cumulants are obtained through the generating function $\chi(\lambda)$ via $\langle\langle n^j\rangle\rangle = (-i)^j \partial_\lambda^j \log \chi|_{\lambda=0}$; (ii) upon proper choice of T_k, an entangled incoming state can generate any value for the Fano factor $F<2$, and (iii) for two spin-1/2 fermions, we show that a simple scattering experiment provides information on the entanglement of the incoming state (cf. also Ref. 18).

Subsequently, we analyze the situation with N fermions and derive the full counting statistics for a constant-voltage (V) drive, thereby generalizing the original result of Levitov and Lesovik[1] to describe transport with an energy-dependent scattering transmission (cf. Ref. 19). Our result,

$$\log \chi_N(\lambda) = N \frac{2\pi\hbar v_{\mathrm{F}}}{eV} \int_0^{eV/\hbar v_{\mathrm{F}}} \frac{dk}{2\pi} \log(1 - T_k + T_k e^{i\lambda}), \tag{5}$$

admits the simple interpretation of the full counting statistics as deriving from the transmission of the unbalanced Fermi sea residing between energies E_F and E_F+eV, with E_F denoting the Fermi energy and V the applied bias. Using an alternative derivation based on Eq. (3) and stationary scattering states, we determine the short-time limit of the counting statistics and rederive binomial result (5) in the long-time situation, with the particle number N replaced by the measuring time t, $N \to teV/2\pi\hbar$. The use of our determinant formula combined with Szegő's theorem[20,21] will allow us to present a rigorous derivation of these results.

In the following, we give a short review of previous work on the subject and then derive characteristic functions (1) and (2) of N incoming fermions. In Sec. III, we apply these results to discuss the statistical transport properties of two fermions. Section IV is devoted to the calculation of the characteristic function for the constant-voltage case starting from N-particle trains and letting the width of the individual wave packets go to infinity. In Sec. V, we derive results (3) and (4) describing the setup involving a time-dependent scattering and counting incoherent superpositions of incoming particles. We rederive the constant-voltage result as an application, including the short-time limit.

II. FULL COUNTING STATISTICS

The first suggestion[22] of a generating function for full counting statistics relied on the straightforward expression $\chi(\lambda,t) = \langle \exp[i\lambda \int dt' \mathcal{I}(t')]\rangle$, where $\mathcal{I}(t)$ denotes the current operator. It then was soon realized[1] that this definition does not correspond to any known (even on the level of a "Gedanken experiment") measuring procedure; still, this first definition produced the correct results for all irreducible zero-frequency current-current correlators $\langle\langle \mathcal{I}_0 \cdots \mathcal{I}_0 \rangle\rangle$ (see also the discussion in Ref. 23). The first "practical" definition[24] of a generating function $\chi(\lambda,t)$, corresponding (at least in principle) to a realistic counting experiment, involved a spin galvanometer as a measurement device (see also Ref. 2). Recently, it has been pointed out[4] that this suggestion (corresponding rather to a Gedanken experiment) could actually be realized with qubits serving as a measuring device, whereby the "environmental noise" generated by the transmitted charge serves as the measurement signal for the full counting statistics. This contrasts with the usual interpretation of the environmental noise as being responsible for the qubit's dephasing[25] expressed through the fidelity and also relates to the competition between the gain of information and dephasing[26] in quantum measurement theory.

The insight on the equivalence between the notions of fidelity and full counting statistics has motivated a first-quantized formalism of the counting problem in terms of wave packets. Fidelity $|\chi_{\mathrm{fid}}|$, the modulus of the overlap $\chi_{\mathrm{fid}} = \langle \Psi_2 | \Psi_1 \rangle$, was introduces by Peres[5] in the context of chaotic systems. It measures the overlap between two wave functions $|\Psi_{1,2}\rangle$ which describe an initial state $|\Psi_0\rangle$ which has evolved under the action of two slightly different Hamiltonians. In the context of full counting statistics of a single particle measured by a spin counter, the wave functions Ψ_1 and Ψ_2 are substituted by scattering states Ψ^+_{out} and Ψ^-_{out} interacting with the spin counter in the states $|\uparrow\rangle$ and $|\downarrow\rangle$, resulting in an expression for the generating function in the form $\chi_1 = \langle \Psi^-_{\mathrm{out}} | \Psi^+_{\mathrm{out}} \rangle$. This first-quantized formulation in terms of wave packets provides a drastic simplification as compared to the original second-quantized formalism.[2] While the use of a second-quantized formalism is mandatory for the description of particles describing bosonic excitations of fields (photons, phonons, etc.), here, we deal with nonrelativistic electrons where the particle number is fixed, thus allowing for an alternative first-quantized description. Moreover, our wave-packet formalism has technical merits (e.g., in the description of energy-dependent scattering or in the classification of two-particle scattering events) and also provides a better physical understanding. We remark, however, that in dealing with finite temperatures we make use of the second-quantized formalism in Fock space.

An alternative method, to the procedure based on a spin counter, was pursued in several contributions[27–29] where the full counting statistics, and—in particular—its generating function $\chi(\lambda,t)$, was constructed using only basic quantum-mechanical definitions; starting with an initial state in the form of an eigenstate of the particle number operator with a fixed particle number to the right of the scatterer (or the "counter"), a second projection (to eigenstates of the number operator) onto the final state is carried out after the observation time t. Both procedures, projection and spin counting, lead to the same expressions for the generating function χ, provided that the incoming state involves no superposition across the scatterer. In the latter situation, the explicit calcu-

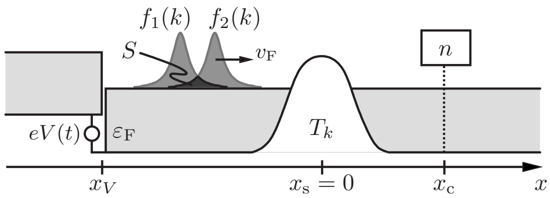

FIG. 1. Quantum wire with scattering center located at x_s giving rise to a momentum-dependent scattering probability T_k. A time-dependent potential $eV(t)$ applied at x_V (to the left of the scatterer) generates incoming wave packets f_1, f_2 with overlap $S=\langle f_2|f_1\rangle$. A counter, placed at x_c (to the right of the scatter), measures the statistics of the number n of transmitted particles. In our analysis, we consider incoming wave packets with momenta $k>0$ residing outside the Fermi sea. As a result, the Fermi sea, which is not accounted for in our analysis, is not disturbed in the asymptotic time limit. For finite measuring times, the presence of the Fermi sea generates additional equilibrium noise which we do not consider in this paper.

lation using a spin counter produces a fidelity describing the decoherence of the spin, while an interpretation in terms of a generating function can produce probabilities for noninteger charge transport[28] and hence is unphysical. On the other hand, the projection method, destroying such a superposition in the course of the first measurement, always admits an interpretation in terms of probabilities.

A. One particle

In this paper, we make extensive use of the first-quantized formulation of the generating function: starting with a simple one-particle problem, we exploit the equivalence between the notion of fidelity and full counting statistics.[4] Consider an incoming wave packet $\psi(x;t\to-\infty)$ from the left of the form

$$\psi(x;t) = \int \frac{dk}{2\pi}\phi_1(k)e^{ikx-i\epsilon(k)t} \tag{6}$$

with normalization $\int(dk/2\pi)|\phi_1(k)|^2=1$ (cf. Fig. 1). In the following, we assume (for simplicity) a linear spectrum $\epsilon = v_F k$ with v_F as the Fermi velocity; at low temperatures and voltages the interesting physics usually takes place near the Fermi points. The momentum $\hbar k$ and the energy $\hbar\epsilon$ are measured with respect to the Fermi momentum k_F and the Fermi energy E_F. Here and below, the wave packets include only momenta with $k>0$ in order not to disturb the Fermi sea which is considered to be the vacuum in our analysis. The scatterer at $x=0$ is characterized by momentum- (energy-) dependent transmission (reflection) amplitudes t_k (r_k; particle reflection takes us to the branch $\epsilon=-v_F k$, with k measured relative to $-k_F$). The spin (or qubit) counter, placed to the right of the scatterer, contributes a phase factor $e^{\pm i\lambda/2}$ to the wave function, where the sign depends on the state $|\uparrow\rangle$, $|\downarrow\rangle$ of the spin counter. The outgoing ($t\to\infty$) wave function assumes the form (we place the counter right behind the scatterer at $x=0$)

$$\psi^{\pm}_{\text{out}}(x;t) = \int \frac{dk}{2\pi}[r_k e^{-ik(x+v_F t)}\Theta(-x) + t_k e^{ik(x-v_F t)}e^{\pm i\lambda/2}\Theta(x)]\phi_1(k) \tag{7}$$

and consists of reflected ($x<0$) and transmitted ($x>0$) parts; $\Theta(x)$ is the unit-step function. The fidelity $\chi_1(\lambda)$ is given by the overlap of wave functions with slightly different perturbations in their evolution; here, with coupling to opposite spin configurations $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$\begin{aligned}\chi_1(\lambda) &= \int dx\,\psi^{-}_{\text{out}}(x;t)^*\psi^{+}_{\text{out}}(x;t)\\ &\overset{(t\to\infty)}{\to} \int \frac{dk}{2\pi}(1-T_k+T_k e^{i\lambda})|\phi_1(k)|^2\\ &= \langle\phi_1|1-\mathcal{T}+\mathcal{T}e^{i\lambda}|\phi_1\rangle,\end{aligned} \tag{8}$$

in the asymptotic or long-time limit, the integration over space is trivially done by exploiting the complete separation of the wave function into transmitted and reflected parts. Furthermore, the time dependence disappears as soon as the transmitted wave function has passed the counter. The transmission probabilities $T_k=|t_k|^2$ are the eigenvalues of the transmission operator $\mathcal{T}=\int(dk/2\pi)T_k|k\rangle\langle k|$. Given the above specific coupling to a spin, the fidelity is equivalent to the characteristic function

$$\chi(\lambda) = \sum_m P_m e^{i\lambda m} \tag{9}$$

of the full counting statistics as defined in Ref. 24, where a spin galvanometer has been used as a measuring device. The Fourier coefficients P_m are the probabilities for transmitting m particles. For the simple example of one incoming particle only two outcomes are possible, particle reflection with probability $P_0=1-\langle\mathcal{T}\rangle$ and particle transmission with $P_1=\langle\mathcal{T}\rangle$, where $\langle\mathcal{T}\rangle=\langle\phi_1|\mathcal{T}|\phi_1\rangle$ denotes the average transmission probability. Knowing the characteristic function, the cumulants $\langle\langle n^j\rangle\rangle$ can be obtained as the coefficients in the Taylor series of $\log\chi(\lambda)$,

$$\langle\langle n^j\rangle\rangle = \left(\frac{d}{id\lambda}\right)^j \log\chi(\lambda)\big|_{\lambda=0}. \tag{10}$$

The ratio $F=\langle\langle n^2\rangle\rangle/\langle n\rangle$ between the second and the first cumulants, called Fano factor, will be of special interest later.

B. *N* particles

Next, we extend the above description to N particles with an incoming wave function $\Psi(\mathbf{k})$ defined in momentum space; the vector $\mathbf{k}=(k_1,\ldots,k_N)$ specifies the N momenta of the particles. We consider independent particles without interaction which scatter independently. After scattering, the outgoing wave function assumes the asymptotic ($t\to\infty$) form,

$$\psi^{\pm}_{\text{out}}(\mathbf{x};t) = \left\{ \prod_{m=1}^{N} \int \frac{dk_m}{2\pi} [r_{k_m} e^{-ik_m(x_m+v_F t)}\Theta(-x_m) + t_{k_m} e^{ik_m(x_m-v_F t)} e^{\pm i\lambda/2}\Theta(x_m)] \right\} \Psi(\mathbf{k}), \quad (11)$$

i.e., the evolution is the product of the single-particle evolutions in expression (7). The characteristic function of the full counting statistics $\chi_N(\lambda)=\int d\mathbf{x}\psi^{-}_{\text{out}}(\mathbf{x};t)^{*}\psi^{+}_{\text{out}}(\mathbf{x};t)$ then can be cast into the form

$$\chi_N(\lambda) = \left\{ \prod_{m=1}^{N} \int \frac{dk_m}{2\pi} (1 - T_{k_m} + T_{k_m} e^{i\lambda}) \right\} |\Psi(\mathbf{k})|^2. \quad (12)$$

So far, we did not specify the specific type of incoming wave function. If we limit ourselves to Slater-determinant states composed of orthonormalized single-particle states ϕ_m,

$$\Psi(k_1, \ldots, k_N) = \frac{1}{\sqrt{N!}} \det \phi_m(k_n), \quad (13)$$

the expression [Eq. (12)] can be rewritten as a single determinant [see Eq. (69)],

$$\chi_N(\lambda) = \det \int \frac{dk}{2\pi} \phi^{*}_m(k)(1 - T_k + T_k e^{i\lambda})\phi_n(k) = \det\langle \phi_m | 1 - \mathcal{T} + \mathcal{T} e^{i\lambda} | \phi_n \rangle, \quad (14)$$

involving the single-particle matrix elements $\langle \phi_m | \mathcal{O} | \phi_n \rangle$ of the operator $\mathcal{O}=1-\mathcal{T}+\mathcal{T}e^{i\lambda}$.

C. Nonorthogonal basis

In a physical realization of such a scattering experiment, one usually does not populate orthogonal states as used in the above construction of the Slater determinant. For example, in the setup of Fig. 1 the electrons typically occupy states f_1 and f_2 with a finite overlap, i.e., they are nonorthogonal. Of course, an N-particle Slater determinant can be constructed as well out of nonorthogonal states $|f_m\rangle$, provided they are linearly independent, i.e., $\det\langle f_m | f_n \rangle \neq 0$. The properly antisymmetrized and normalized wave function (13) then acquires the form

$$\Psi^{f}(k_1, \ldots, k_N) = \frac{1}{\sqrt{N!\ \det\langle f_m | f_n \rangle}} \det f_m(k_n). \quad (15)$$

Inserting this expression into Eq. (12) and repeating the calculation that led to Eq. (14), we obtain the generating function in the form of a ratio of two determinants,

$$\chi_N(\lambda) = \frac{\det\langle f_m | 1 - \mathcal{T} + \mathcal{T} e^{i\lambda} | f_n \rangle}{\det\langle f_m | f_n \rangle} = \frac{\det(\mathbf{S}^f - \mathbf{T}^f + \mathbf{T}^f e^{i\lambda})}{\det \mathbf{S}^f} \quad (16)$$

with the two $N \times N$ matrices,

$$\mathbf{S}^f_{mn} = \langle f_m | f_n \rangle, \quad \mathbf{T}^f_{mn} = \langle f_m | \mathcal{T} | f_n \rangle. \quad (17)$$

D. Invariance of Slater determinants under linear transformations

It turns out that expression (16) for the generating function can be drastically simplified and rewritten in a generalized binomial form. As a first step toward this goal, one has to realize that an N-dimensional Hilbert space H_N, spanned by the single-particle wave functions $f_n(k)$, defines exactly one properly antisymmetrized wave function or, equivalently, there exists (up to a phase factor) only one associated N-particle Slater-determinant state. The antisymmetrized N-particle state is thus a property of the Hilbert space H_N and is independent of the basis chosen.[30]

Consider, as a simple example, a two-particle Slater-determinant state (in second-quantized notation) $|\Psi\rangle = a_2^{\dagger} a_1^{\dagger} |0\rangle$, with the vacuum state $|0\rangle$ and fermionic operators $a_{1,2}$. Defining the new operators $a_{\pm}=(a_1 \pm a_2)/\sqrt{2}$, we easily see that the two-particle state

$$a_+^{\dagger} a_-^{\dagger} |0\rangle = \frac{1}{2}(a_1^{\dagger} + a_2^{\dagger})(a_1^{\dagger} - a_2^{\dagger})|0\rangle = a_2^{\dagger} a_1^{\dagger} |0\rangle = |\Psi\rangle \quad (18)$$

remains unchanged. Consider then a general N-particle Slater-determinant state of form (15). Transforming the basis states $f_m(k)$ to new states $g_m(k)$ via the complex linear transformation,

$$g_m(k) = \sum_n \mathbf{A}_{nm} f_n(k), \quad \det \mathbf{A} \neq 0, \quad (19)$$

the antisymmetric combination,

$$\det g_m(k_n) = (\det \mathbf{A}) \det f_m(k_n), \quad (20)$$

remains invariant up to the factor $\det \mathbf{A}$; here, we have used the fact that the determinant of the product of two matrices is the product of the individual determinants. Furthermore, the normalized N-particle Slater-determinant states Ψ^f and Ψ^g obey the relation

$$\Psi^{g}(k_1, \ldots, k_N) = \text{sgn}(\det \mathbf{A}) \Psi^{f}(k_1, \ldots, k_N) \quad (21)$$

with $\text{sgn}(x)=x/|x|$. The only effect of adopting a new basis is the appearance of an overall phase factor sgn(det $\mathbf{A}$) which drops out of characteristic function (12). Therefore, the full counting statistics calculated in the bases f and g give identical results.

E. Diagonalization

The above invariance can be used to simplify the calculation of the full counting statistics. Furthermore, even without specification of the (time-independent) scatterer, one can obtain valuable insights about the structure of possible outcomes in the counting statistics. In particular, it turns out that the most general full counting statistics for a Slater-determinant state is given by a generalized binomial expression of form (2).

Let us first investigate how the invariance under linear transformations [Eq. (19)] manifests itself in the determinant formula [Eq. (16)]. To this end, we note that any single-particle matrix $\mathbf{B}$ of form (17) transforms under the linear transformation $\mathbf{A}$ of the basis functions according to

$$\mathbf{B}^g = \mathbf{A}^\dagger \mathbf{B}^f \mathbf{A}, \quad \mathbf{B} = \mathbf{S}, \mathbf{T}. \tag{22}$$

Since $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$, we find that the characteristic function χ_N (we define $\mathbf{X}^f = \mathbf{S}^f - \mathbf{T}^f + \mathbf{T}^f e^{i\lambda}$) is invariant under the change in basis,

$$\chi_N = \frac{\det \mathbf{X}^f}{\det \mathbf{S}^f} = \frac{|\det \mathbf{A}|^2 \det \mathbf{X}^f}{|\det \mathbf{A}|^2 \det \mathbf{S}^f} = \frac{\det \mathbf{X}^g}{\det \mathbf{S}^g}. \tag{23}$$

This invariance can be exploited by going over to new orthogonal basis functions $g_m(k)$ with an overlap matrix $\mathbf{S}^g_{mn} = \delta_{mn}$ and a transmission matrix assuming a diagonal form $\mathbf{T}^g_{mn} = \tau_m \delta_{mn}$. The possibility of simultaneous diagonalization of the matrices $\mathbf{T}^g_{mn}$ and $\mathbf{S}^g_{mn}$ is a consequence of transformation law (22), characteristic of bilinear forms (as opposed to linear transformations $\mathbf{L}$ which transform according to $\mathbf{L}^g = \mathbf{A}^{-1}\mathbf{L}^f\mathbf{A}$), combined with the positivity of $\mathbf{S}^f$. The corresponding eigenbasis g_m and eigenvalues τ_m of $\mathbf{T}^g_{mn}$ can be found by solving the generalized eigenvalue problem,

$$(\mathbf{T}^f - \tau_m \mathbf{S}^f) a_m = 0, \tag{24}$$

with the normalization $a_m^\dagger \mathbf{S}^f a_m = 1$.[31] The eigenvectors a_m constitute the column vectors of the transformation matrix $\mathbf{A} = (a_1, \ldots, a_N)$. The eigenvalues are given by the roots of the characteristic polynomial $\det(\mathbf{T}^f - \tau \mathbf{S}^f) = 0$. The full counting statistics [Eq. (16)], written in the new basis $g_m(k)$, assumes the generalized binomial form,

$$\chi_N(\lambda) = \prod_{m=1}^{N} (1 - \tau_m + \tau_m e^{i\lambda}), \tag{25}$$

where the determinant has been evaluated explicitly and the result depends only on the eigenvalues τ_m. The generalized eigenvalue problem can be reduced to a normal one by rewriting the problem in a orthogonalized basis $\phi_m(k)$, with $\mathbf{S}^\phi = \mathbb{1}_N$, which can be obtained by the Gram-Schmidt procedure or by setting $\phi_m(k) = \Sigma_n [(\mathbf{S}^f)^{-1/2}]_{nm} f_n(k)$.

From the above, we see that the concrete form of eigenvalue problem (24) is basis dependent, whereas the eigenvalues and vectors are simply a property of the transmission operator $\mathcal{T}$ operating in the Hilbert space H_N with the scalar product $\langle f | g \rangle$. Indeed, it is possible to find the eigenvalues and eigenvectors in a basis independent way using the positive-definite quadratic forms $T(g) = \langle g | \mathcal{T} | g \rangle$ and $S(g) = \langle g | g \rangle$, $g \in H_N$. Representing the bilinear form $T(g)$ with fixed $S(g) = 1$ as a polar plot with $T(g)$ as the radius and g defining the direction in H_N, we obtain an ellipsoid in N-dimensional space. The lengths of the main axes of this ellipsoid then constitute the eigenvalues and the associated directions of the eigenvectors of problem (24).[32] The eigenvalues τ_m are constrained to the interval $[0,1]$ as $T(g) \geq 0$ and $T(g) \leq S(g)$ due to unitarity.

F. Full counting statistics for entangled states

The above discussion has concentrated on incoming states described by a single Slater determinant, i.e., nonentangled states with Slater rank 1. It is instructive to generalize this discussion to entangled states involving a coherent superposition of Slater determinants. We start from an incoming state of N particles with Slater rank 2,

$$\Psi(\mathbf{k}) = \alpha \Psi^{\mathrm{I}}(\mathbf{k}) + \beta \Psi^{\mathrm{II}}(\mathbf{k}), \tag{26}$$

where $\Psi^{\mathrm{I}}(\mathbf{k})$ and $\Psi^{\mathrm{II}}(\mathbf{k})$ are normalized N-particle Slater determinants describing particles incoming from the left and made from single-particle states $f^{\mathrm{I}}_m(k)$ and $f^{\mathrm{II}}_m(k)$, $m = 1, \ldots, N$; the complex numbers α and β have been chosen such as to make $\Psi(\mathbf{k})$ normalized. The characteristic function for full counting statistics (12) assumes the form

$$\chi_N(\lambda) = \left[\prod_{m=1}^{N} \int \frac{dk_m}{2\pi} (1 - T_{k_m} + T_{k_m} e^{i\lambda}) \right] \Big[|\alpha|^2 |\Psi^{\mathrm{I}}(\mathbf{k})|^2 + |\beta|^2 |\Psi^{\mathrm{II}}(\mathbf{k})|^2 + 2\,\mathrm{Re}\{\alpha\beta^* \Psi^{\mathrm{I}}(\mathbf{k}) \Psi^{\mathrm{II}}(\mathbf{k})^*\} \Big], \tag{27}$$

where Re denotes the real part. The first two terms reduce to generating functions for simple Slater-determinant states and we can write

$$\chi_N(\lambda) = |\alpha|^2 \chi^{\mathrm{I}}_N(\lambda) + |\beta|^2 \chi^{\mathrm{II}}_N(\lambda) + \alpha\beta^* \chi^{\mathrm{mix}}_N(\lambda) + \alpha^* \beta \chi^{\mathrm{mix}}_N(-\lambda)^* \tag{28}$$

with

$$\chi^{\mathrm{I}}_N(\lambda) = \frac{\det(\mathbf{S}^{f^{\mathrm{I}}} - \mathbf{T}^{f^{\mathrm{I}}} + \mathbf{T}^{f^{\mathrm{I}}} e^{i\lambda})}{\det \mathbf{S}^{f^{\mathrm{I}}}},$$

$$\chi^{\mathrm{II}}_N(\lambda) = \frac{\det(\mathbf{S}^{f^{\mathrm{II}}} - \mathbf{T}^{f^{\mathrm{II}}} + \mathbf{T}^{f^{\mathrm{II}}} e^{i\lambda})}{\det \mathbf{S}^{f^{\mathrm{II}}}},$$

$$\chi^{\mathrm{mix}}_N(\lambda) = \frac{\det(\mathbf{S}^{\mathrm{mix}} - \mathbf{T}^{\mathrm{mix}} + \mathbf{T}^{\mathrm{mix}} e^{i\lambda})}{\sqrt{\det \mathbf{S}^{f^{\mathrm{I}}} \mathbf{S}^{f^{\mathrm{II}}}}}. \tag{29}$$

The matrices with superscripts f^{I} and f^{II} have been defined in Eq. (17), while the new Hermitian matrices with a superscript "mix" are given by the mixed matrix elements,

$$\mathbf{S}^{\mathrm{mix}}_{mn} = \langle f^{\mathrm{II}}_m | f^{\mathrm{I}}_n \rangle, \quad \mathbf{T}^{\mathrm{mix}}_{mn} = \langle f^{\mathrm{II}}_m | \mathcal{T} | f^{\mathrm{I}}_n \rangle. \tag{30}$$

The first two terms in Eq. (28) can be diagonalized as before [cf. Eq. (24)],

$$\chi^{\mathrm{I}}_N(\lambda) = \prod_{m=1}^{N} (1 - \tau^{\mathrm{I}}_m + \tau^{\mathrm{I}}_m e^{i\lambda}), \tag{31}$$

$$\chi^{\mathrm{II}}_N(\lambda) = \prod_{m=1}^{N} (1 - \tau^{\mathrm{II}}_m + \tau^{\mathrm{II}}_m e^{i\lambda}), \tag{32}$$

with the eigenvalues τ^{I}_m and τ^{II}_m given by the roots of $\det(\mathbf{T}^{\mathrm{I}} - \tau^{\mathrm{I}} \mathbf{S}^{\mathrm{I}}) = 0$ and $\det(\mathbf{T}^{\mathrm{II}} - \tau^{\mathrm{II}} \mathbf{S}^{\mathrm{II}}) = 0$.

Let us then concentrate on the characteristic function $\chi^{\mathrm{mix}}(\lambda)$. Unfortunately, there is no generic procedure to follow in this case, as the matrices $\mathbf{S}^{\mathrm{mix}}$ and $\mathbf{T}^{\mathrm{mix}}$ are not Hermitian anymore and hence expression (27) cannot be further simplified in general. In particular, the characteristic function $\chi^{\mathrm{mix}}(\lambda)$ is not invariant under individual transformations of the bases f^{I}_m and f^{II}_m (such basis transformations leave the Slater determinants invariant only up to a phase factor, which dropped out in the calculation of the characteristic function of a single Slater-determinant state but does not when two

Slater determinants are superimposed coherently). In order to proceed further, we restrict ourselves to specific situations where $\mathbf{S}^{\rm mix}=0$ or $\det \mathbf{S}^{\rm mix}\neq 0$. The most trivial case is realized for mutually orthogonal sets of basis functions $f_m^{\rm I}$ and $f_m^{\rm II}$ where $\mathbf{S}^{\rm mix}=0$; if, in addition, $\det \mathbf{T}^{\rm mix}=0$, we have $\chi^{\rm mix}(\lambda)=0$ (see also Sec. III C below), else $\chi^{\rm mix}(\lambda)=\tau^{\rm mix}(e^{i\lambda}-1)^N$ with $\tau^{\rm mix}=\det \mathbf{T}^{\rm mix}/\sqrt{\det \mathbf{S}^{f^{\rm I}}\mathbf{S}^{f^{\rm II}}}$.

Second, let us assume that $\mathbf{S}^{\rm mix}$ is invertible, $\det \mathbf{S}^{\rm mix}\neq 0$. Let $\tau_m^{\rm mix}$ be the roots of the polynomial,

$$\det[\mathbf{T}^{\rm mix}-\tau^{\rm mix}\mathbf{S}^{\rm mix}]=0. \quad (33)$$

The matrix $\mathbf{T}^{\rm mix}(\mathbf{S}^{\rm mix})^{-1}$ then can be brought into a Jordan canonical form with $\tau_m^{\rm mix}$ on the diagonal and the characteristic function assumes the simple form,

$$\chi_N^{\rm mix}(\lambda)=\frac{\det \mathbf{S}^{\rm mix}}{\sqrt{\det \mathbf{S}^{f^{\rm I}}\mathbf{S}^{f^{\rm II}}}}\prod_{m=1}^{N}(1-\tau_m^{\rm mix}+\tau_m^{\rm mix}e^{i\lambda}). \quad (34)$$

The procedure outlined above is straightforwardly generalized to states with higher Slater rank.

III. TWO PARTICLES

A. Full counting statistics

The above findings have interesting generic consequences for the charge transport of fermionic particles; in the following, we discuss the simplest case of two particles (see Fig. 1), where nontrivial exchange properties manifest themselves. For $N=2$ particles diagonalization (24) can be carried out explicitly for arbitrary matrices $\mathbf{T}^f$ and $\mathbf{S}^f$. The two eigenvalues $\tau_{1,2}$ are given by

$$\tau_{1,2}=\frac{\alpha\mp\sqrt{\alpha^2-\det \mathbf{T}^f\det \mathbf{S}^f}}{\det \mathbf{S}^f}, \quad (35)$$

where the parameter $2\alpha=\mathbf{S}_{22}^f\mathbf{T}_{11}^f+\mathbf{S}_{11}^f\mathbf{T}_{22}^f-2\,\mathrm{Re}(\mathbf{S}_{12}^f\mathbf{T}_{21}^f)$. Alternatively, the eigenvalues $0\leq\tau_m\leq 1$ are given by a minimum or maximum property,[32]

$$\tau_1=\min_{g\in H_2|S(g)=1}T(g),\quad \tau_2=\max_{g\in H_2|S(g)=1}T(g), \quad (36)$$

with the eigenvectors $g_{1,2}(k)$ given by those functions where the minimum or maximum values are attained, i.e., $T(g_{1,2})=\tau_{1,2}$. Once the eigenvalues τ_m are known, the characteristic function χ_2 assumes the simple generalized binomial form,

$$\chi_2(\lambda)=(1-\tau_1+\tau_1e^{i\lambda})(1-\tau_2+\tau_2e^{i\lambda}). \quad (37)$$

As a result, we find that in the new basis g_m, the two particles traverse the scatterer independent of one another, i.e., the characteristic function is a simple product of independent one-particle characteristic functions. Even more, the characteristic function is determined by the Hilbert space spanned by the incoming states $f_{1,2}$ and is independent of the choice of basis. Exchange effects manifest themselves when comparing result (37) for the Slater determinant $\Psi^f\propto\det f_m(k_n)$ with the result $\chi_2^{\rm dist}(\lambda)=(1-\mathbf{T}_{11}^f+\mathbf{T}_{11}^fe^{i\lambda})(1-\mathbf{T}_{22}^f+\mathbf{T}_{22}^fe^{i\lambda})$ for distinguishable particles, $\Psi^{\rm dist}\propto f_1(k_1)f_2(k_2)$; exchange effects are absent if both matrix elements $\mathbf{S}_{21}^f=\langle f_2|f_1\rangle=0$ and $\mathbf{T}_{21}^f=0$, i.e., for orthogonal initial and transmitted states. On the other hand, a finite overlap of at least one pair of these states generates finite exchange effects via the substitution of $\mathbf{T}_{mm}^f$ in $\chi_2^{\rm dist}$ by the eigenvalues τ_m in χ_2.

The minimum or maximum property described above entails a set of *a priori* inequalities for the transmission probabilities P_n involving the transmission matrix elements $T_{\rm min}=\min\{\mathbf{T}_{11}^f,\mathbf{T}_{22}^f\}$ and $T_{\rm max}=\max\{\mathbf{T}_{11}^f,\mathbf{T}_{22}^f\}$; note that while the probabilities P_n do account for exchange effects, the single-particle matrix elements $\mathbf{T}_{mm}^f$ obviously do not. With initial (nonorthogonal) wave packets f_m normalized to unity, $S(f_m)=1$, the search for the extrema in Eq. (36) includes these states as well. We then obtain the set of inequalities $0\leq\tau_1\leq T_{\rm min}\leq T_{\rm max}\leq\tau_2\leq 1$. Using them to estimate $P_0=(1-\tau_1)(1-\tau_2)\leq(1-\tau_2)$, $P_2=\tau_1\tau_2\leq\tau_1$, and $P_1=1-P_0-P_2$, we can derive the following bounds,

$$P_0\leq 1-T_{\rm max},\quad P_1\geq T_{\rm max}-T_{\rm min},\quad P_2\leq T_{\rm min}, \quad (38)$$

for the *transmission probabilities for two particles*. The above bounds set an upper limit on bunching (P_2 and P_0) and a lower limit on antibunching (P_1). Note though that the bound on P_2 does not exclude an increase (due to exchange) in the transmission probability beyond the "classical" value $P_2^{\rm dist}=\mathbf{T}_{11}^f\mathbf{T}_{22}^f$ for distinguishable particles (see $\chi_2^{\rm dist}$ above). Indeed, since $\mathbf{T}_{11}^f\mathbf{T}_{22}^f\leq T_{\rm min}$, a value $P_2\gg\mathbf{T}_{11}^f\mathbf{T}_{22}^f$ remains possible. Such a result has been recently observed;[33] the probability of two-electron events in the electron emission from a Cs_3Sb photocathode in a photomultiplier tube has been found to be much larger than the square of the probability for single-electron emission. This was observed both in the case of thermal emission without photocathode illumination and photoemission under weak photocathode illumination. Furthermore, as detailed calculation shows, a large P_2 can also be obtained for wave packets with amplitudes $f_2(k)=f_1(k+\delta k)$ shifted in k space and a large overlap integral $\mathbf{S}_{21}^f$, combined with a transmission amplitude suppressing k values in the overlap region.

B. Restrictions due to binomial statistics

An arbitrary two-particle scattering process is fully characterized by the three parameters P_0, P_1, P_2, from which only two are independent; here, we assume that we can transmit only integer charges (no charge fractionalization). In Figs. 2(a) and 2(b), we find the regions with different statistical properties that can be generated in a two-fermion scattering process, both in P_0-P_2 parameter space as well as in the noise $\langle\langle n^2\rangle\rangle$ versus average number $\langle n\rangle$ diagram. We start with the definition of the physically accessible regime in these diagrams: requiring that $P_1=1-P_0-P_2\geq 0$ [Fig. 2(a)] and $P_0,P_1,P_2\geq 0$ [Fig. 2(b)], we find that the black regions are forbidden.

Traditionally, starting from Poissonian statistics ($F=1$) relevant for the coherent light emitted from a laser or for the transport of a classical electron gas in a vacuum tube, much emphasis has been put on the distinction between sub- and super-Poissonian statistics, with reduced and enhanced noise intensities as quantified by Fano factors $F<1$ (sub-Poissonian noise) and $F>1$ (super-Poissonian processes). It

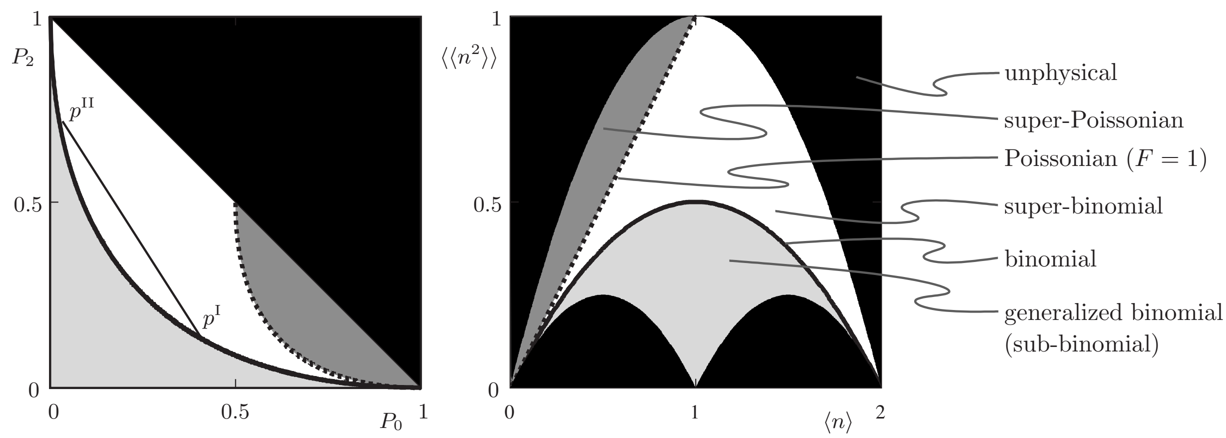

FIG. 2. Diagrams describing the generic statistical properties of two-particle transmission on the left as a P_2-P_0 probability diagram and on the right as a noise-charge $\langle\langle n^2\rangle\rangle$-$\langle n\rangle$ diagram. The black regions are unphysical with probabilities P_0, P_1, P_2 residing outside [0,1]. The light gray regions describe generalized binomial (sub-binomial) processes [Eq. (37)] bounded by the black line characterizing usual binomial processes. The dotted lines correspond to a Fano factor $F=\langle\langle n^2\rangle\rangle/\langle n\rangle$ equal to one. Within the dark-gray regions noise is super-Poissonian with a Fano factor $F>1$. Note that in order to observe super-Poissonian noise the reflection probability has to be large, such that $P_0>1/2$ and $\langle n\rangle<1$.

appears to us that in the context of degenerate fermions, the generic starting point is the binomial statistics, instead, and more relevant qualifications are given by the regimes of sub-binomial and superbinomial processes that will be introduced below.

Nevertheless, let us start our analysis with the traditional classification comparing a process with Poissonian statistics, which is realized on the dotted line in Fig. 2(a) defined through the relation

$$F = \frac{P_0(1-P_0)+P_2(1-P_2)+2P_0P_2}{1-P_0+P_2} = 1, \tag{39}$$

i.e.,

$$P_2 = P_0 - \sqrt{2P_0 - 1}, \quad P_0 \geq 1/2. \tag{40}$$

Within the dark-gray region noise is super-Poissonian, which is usually associated with the bunching of particles and therefore with bosonic statistics. Note that Fano factors larger than the Poissonian value 1 require a large reflection probability $P_0>1/2$; only when most of the particles are reflected one can observe the "bunching" of the remaining transmitted objects.

A much more natural classification for our fermion system is in terms of (deviations from) binomial statistics. The characteristic function χ_2 for two fermions in a Slater-determinant state can be cast into the generalized binomial form [Eq. (37)], which depends on two parameters τ_1, τ_2. As a consequence, the probabilities satisfy the additional inequality,

$$\sqrt{P_0} + \sqrt{P_2} \leq 1. \tag{41}$$

This condition follows from expressing the parameters τ_1, τ_2 through the probabilities P_0, P_2 using the relations $P_0=(1-\tau_1)(1-\tau_2)$ and $P_2=\tau_1\tau_2$; requiring a positive discriminant of the resulting (quadratic) equation implies constraint (41) which defines the light gray region in Fig. 2(a), naturally termed as the "sub-binomial" regime. The (thick) black line bounding the general binomial (or sub-binomial) region is the line of usual binomial statistics, which is realized for the case of degenerate transmission coefficients $\tau_1=\tau_2$ as they appear if the scattering does not depend on energy.

The region with super-Poissonian noise (dark gray) and the sub-binomial region (light gray) are distinct, with the statistics of fermions incoming in a Slater-determinant state always residing in the sub-binomial domain. Note that the counting statistics of an arbitrary two-particle process (without specification of exchange properties) also depends on two out of the three parameters P_0, P_1, P_2 (as the constraint $P_0+P_1+P_2=1$ needs to be fulfilled) but cannot be cast into form (37) in general; hence these processes are devoid of such an additional restriction.

The P_2-P_0 diagram can be transcribed to the (experimentally more relevant) $\langle\langle n^2\rangle\rangle$-$\langle n\rangle$ diagram [cf. Fig. 2(b)]. The physical constraints $0\leq P_0,P_1,P_2$ lead to the set of inequalities,

$$\langle\langle n^2\rangle\rangle \geq \langle n\rangle(1-\langle n\rangle),$$

$$\langle\langle n^2\rangle\rangle \leq \langle n\rangle(2-\langle n\rangle),$$

$$\langle\langle n^2\rangle\rangle \geq (\langle n\rangle - 1)(2-\langle n\rangle), \tag{42}$$

which can be cast into the more compact form $(m+1-\langle n\rangle)(\langle n\rangle-m)\leq\langle\langle n^2\rangle\rangle\leq\langle n\rangle(2-\langle n\rangle)$, with $m=0,1$. The single large and two small parabolas bounding the unphysical (black) regions are given by the second and the two (for $m=0,1$) first inequalities. For the generalized (or sub-)binomial statistics, the additional constraint assumes the form

$$F = \frac{\langle\langle n^2 \rangle\rangle}{\langle n \rangle} \le 1 - \langle n \rangle /2, \tag{43}$$

with the equality applying to the binomial case with $\tau_1 = \tau_2$. Within the gray region of the diagram the noise is sub-binomial, $F \le 1 - \langle n \rangle/2$, and hence trivially sub-Poissonian, $F \le 1$. Note that noiseless transmission of charge requires that an integer average charge is transmitted.

The generalization of the above analysis to N incoming particles in a Slater-determinant state is straightforward. The generalized binomial characteristic function is given by Eq. (2). The positivity of the probabilities $P_m \ge 0$, $m=0,\ldots,N$, imposes the $N+1$ restrictions on the first two momenta $\langle n \rangle$ and $\langle\langle n^2 \rangle\rangle$, $(m+1-\langle n \rangle)(\langle n \rangle - m) \le \langle\langle n^2 \rangle\rangle \le \langle n \rangle (N - \langle n \rangle)$, with $m=0,\ldots,N-1$, defining a simple generalization of Fig. 2(b) with one large and N small parabolas. In the generalized binomial case, the additional constraint,

$$F = \frac{\langle\langle n^2 \rangle\rangle}{\langle n \rangle} \le 1 - \langle n \rangle / N \le 1, \tag{44}$$

tells that the incoming Slater-determinant states produce a sub-binomial noise statistics. A similar result was found recently[34] in the context of adiabatic pumping. The authors considered a time-dependent scattering matrix in the instant scattering approximation (i.e., an energy-independent scatterer) and obtained a generating function in a product form describing a generalized binomial statistics with parameters $u_m \le 0$; the u_m relate to our τ_m via $\tau_m = (1-u_m)^{-1}$.

C. Entangled states

The above discussion for two particles lets us conclude that incoming Slater-determinant states generate Fano factors $F \le 1 - \langle n \rangle /2 \le 1$; such states are nonentangled. On the other hand, an entangled two-particle state can be generated with a sum of two Slater determinants; such an entangled state (with $0 < \alpha < 1$, i.e., a state with Slater rank 2, see Sec. II F),

$$\Psi(k_1,k_2) = \sqrt{\alpha}\Psi^{\mathrm{I}}(k_1,k_2) + \sqrt{1-\alpha}\Psi^{\mathrm{II}}(k_1,k_2), \tag{45}$$

is sufficient to generate all possible types of two-particle statistics. We choose the Slater-determinant wave functions Ψ^{I} and Ψ^{II} (incoming from the left) such that they occupy different parts of momentum space, e.g., Ψ^{I} has only components below k_c and Ψ^{II} above. Furthermore, let the transmission be $T_1 = T_{k<k_c}$ below k_c and $T_2 = T_{k>k_c}$ above. For such a setup, all the overlap integrals vanish, e.g., $\int (dk_1 dk_2 / 4\pi^2) \Psi^{\mathrm{II}*}(k_1,k_2)\Psi^{\mathrm{I}}(k_1,k_2) = 0$, and we obtain [cf. Eq. (28)]

$$\chi_2(\lambda) = \alpha(1 - T_1 + T_1 e^{i\lambda})^2 + (1-\alpha)(1 - T_2 + T_2 e^{i\lambda})^2, \tag{46}$$

that is, the generating function is simply the weighted sum of the two individual generating functions for the Slater-determinant states. The statistics of such entangled wave functions is described by points in the P_2-P_0 diagram of Fig. 2(a) which lie on a straight line between the point p^{I} for Ψ^{I} and the point p^{II} for Ψ^{II} with α parametrizing the line. Both p^{I} and p^{II} are situated on the binomial line, while the line connecting them may enter the superbinomial or even the super-Poissonian region; for example, setting $T_1 = 0$ and $T_2 = 1$, the characteristic function is given by $\chi_2 = \alpha + (1-\alpha)e^{2i\lambda}$ and $F = 2\alpha$, which assumes values between zero and two (note that in the limit $\alpha \to 1$, wave function (45) is of Slater rank 1, but nevertheless, the Fano factor approaches $F=2$. As the Fano factor for $P_0 = 1$ assumes the form 0/0 its value depends on the direction from which $P_0 = 1$ is approached). As simple Slater determinants produce only Fano factors up to $1 - \langle n \rangle /2$, a larger value serves as a test for the entanglement of the two particles.[35,36] For N incoming particles in an entangled state of rank 2, the analogous construction (cf. Sec. II F) produces a Fano factor $F = N\alpha$ with $0 < \alpha < 1$, i.e., super-Poissonian statistics can be admitted for sufficiently large α.

D. Two spin-1/2 particles

Next, we consider the situation in the setup of Fig. 1 with incoming particles in normalized states $f_1(k)$ and $f_2(k)$ with overlap $S = \mathbf{S}^f_{21} = \langle f_2 | f_1 \rangle$ and carrying a spin-1/2 degree of freedom. We consider the case of spin-independent scattering, hence the coefficients in $\mathbf{T}^f$ depend exclusively on $f_1(k)$ and $f_2(k)$. The four properly symmetrized states available to the two incoming particles are denoted by $\Psi_{s,m_s}(\mathbf{k})$, with $s=0$ as the singlet ($m_s = 0$) state and $s=1$ as the three ($m_s = -1, 0, +1$) triplet states. The degrees of freedom $\mathbf{k}$ involve the momenta k_m and spins s_m of the particles, $\mathbf{k} = (k_1, s_1; k_2, s_2)$. The triplet states with $m_s = \pm 1$ are simple Slater-determinant states,

$$\Psi_{1,\pm 1}(\mathbf{k}) = \frac{1}{\sqrt{2(1-|S|^2)}}[f_1(k_1)\chi_{\uparrow/\downarrow}(s_1) f_2(k_2)\chi_{\uparrow/\downarrow}(s_2) - [(k_1,s_1) \leftrightarrow (k_2,s_2)]]. \tag{47}$$

The characteristic function χ_2 for the full counting statistics then is of the generalized binomial form with $\tau_{1/2}$ given by Eq. (35),

$$\begin{aligned}\chi_{1,\pm 1}(\lambda) &= (1 - \tau_1 + \tau_1 e^{i\lambda})(1 - \tau_2 + \tau_2 e^{i\lambda}) \\ &= \frac{(1 - \mathbf{T}^f_{11} + \mathbf{T}^f_{11} e^{i\lambda})(1 - \mathbf{T}^f_{22} + \mathbf{T}^f_{22} e^{i\lambda})}{1 - |S|^2} \\ &\quad - \frac{(S - \mathbf{T}^f_{21} + \mathbf{T}^f_{21} e^{i\lambda})(S^* - \mathbf{T}^f_{12} + \mathbf{T}^f_{12} e^{i\lambda})}{1 - |S|^2}.\end{aligned} \tag{48}$$

The states with $m_s = 0$ are more interesting as they are of Slater rank 2. Defining

$$f_1^{\mathrm{I}}(k,s) = f_1(k)\chi_\uparrow(s), \quad f_2^{\mathrm{I}}(k,s) = f_2(k)\chi_\downarrow(s),$$

$$f_1^{\mathrm{II}}(k,s) = f_1(k)\chi_\downarrow(s), \quad f_2^{\mathrm{II}}(k,s) = f_2(k)\chi_\uparrow(s), \tag{49}$$

we have

$$\Psi_{0/1,0}(\mathbf{k}) = \frac{1}{\sqrt{2(1 \pm |S|^2)}}[\Psi^{\mathrm{I}}(\mathbf{k}) \mp \Psi^{\mathrm{II}}(\mathbf{k})] \tag{50}$$

with $\Psi^{\mathrm{I/II}}(\mathbf{k})$ as the normalized two-particle Slater determinants made from the states $f_m^{\mathrm{I/II}}$. The calculation of the characteristic function follows the procedure outlined above: as the matrices $\mathsf{T}^{\mathrm{I/II}}_{mn}=\mathsf{T}^{f}_{mm}\delta_{mn}$ and $\mathsf{S}^{\mathrm{I/II}}_{mn}=\delta_{mn}$ are diagonal (particles 1 and 2 are distinguishable), we immediately have

$$\chi^{\mathrm{I/II}}(\lambda)=(1-\mathsf{T}^{f}_{11}+\mathsf{T}^{f}_{11}e^{i\lambda})(1-\mathsf{T}^{f}_{22}+\mathsf{T}^{f}_{22}e^{i\lambda}). \qquad (51)$$

For the calculation of $\chi^{\mathrm{mix}}(\lambda)$, the matrices $\mathsf{T}^{\mathrm{mix}}$ and $\mathsf{S}^{\mathrm{mix}}$ need to be evaluated. In the present case, they are purely off diagonal with the off-diagonal matrix element given by $\mathsf{T}^{\mathrm{mix}}_{21}=\mathsf{T}^{f}_{21}$ and $\mathsf{S}^{\mathrm{mix}}_{21}=S$. Calculating the determinants in Eq. (29), we obtain the mixed component in the form

$$\chi^{\mathrm{mix}}(\lambda)=-(S-\mathsf{T}^{f}_{21}+\mathsf{T}^{f}_{21}e^{i\lambda})(S^{*}-\mathsf{T}^{f}_{12}+\mathsf{T}^{f}_{12}e^{i\lambda}) \qquad (52)$$

and the characteristic function is given by

$$\chi_{0/1,0}(\lambda)=\frac{(1-\mathsf{T}^{f}_{11}+\mathsf{T}^{f}_{11}e^{i\lambda})(1-\mathsf{T}^{f}_{22}+\mathsf{T}^{f}_{22}e^{i\lambda})}{1\pm|S|^2} \pm\frac{(S-\mathsf{T}^{f}_{21}+\mathsf{T}^{f}_{21}e^{i\lambda})(S^{*}-\mathsf{T}^{f}_{12}+\mathsf{T}^{f}_{12}e^{i\lambda})}{1\pm|S|^2}. \qquad (53)$$

Result (53) agrees with the results in Ref. 12. The characteristic functions for the two spin triplet states $s=1$ with maximal magnetization $m_s=\pm 1$ and the characteristic function for the triplet $s=1$, $m_s=0$ with zero magnetization coincide with the one for a Slater determinant of spinless fermions [Eq. (37)]. This is because all three states involve identical orbital wave functions and the scattering process does not depend on the spin part of the wave function. The corresponding average number of particles $\langle n\rangle_{1,m_s}$ and noise $\langle\langle n^2\rangle\rangle_{1,m_s}$ reside within the region of generalized binomial statistics (cf. Fig. 2),

$$F_{1,m_s}\le 1-\langle n\rangle_{1,m_s}/2. \qquad (54)$$

The entangled singlet state (with $s=0$) does not necessarily fulfill this condition. Rather opposite, for the case where the individual transmission probabilities of the two particles are equal, $\mathsf{T}^{f}_{11}=\mathsf{T}^{f}_{22}$, the moments and the Fano factor always reside outside the region allowed by the generalized binomial statistics,

$$F_{0,0}\ge 1-\langle n\rangle_{0,0}/2, \qquad (55)$$

as a lengthy but straightforward calculation shows. Hence, this rather trivial setup can be used to discriminate singlet from triplet states and also serves as an indicator of entanglement (as long as the inequalities are strict which is the case as long as $S\neq 0$ and $\mathsf{T}^{f}_{21}\neq S\mathsf{T}^{f}_{11}$).

A similar experiment was proposed by Burkard *et al.*,[18] which had two particles with equal energy come in from different arms in a symmetric beam splitter (see Ref. 14 for a calculation of the full counting statistics for this setup). Our setup involves one single lead only at the expense of requiring an energy-dependent transmission probability (otherwise we end up on the binomial line which is devoid of any separation power). Furthermore, the discrimination between singlet and triplet states is determined by the presence or absence of generalized binomial statistics and hence involves the binomial bound $1-\langle n\rangle_{0,0}/2$ on F.

IV. *N*-PARTICLE TRAINS

We consider the case of N incoming particles, all with the same shape of the wave function $f(k)$ aligned regularly in real space with separation a; the wave function of the mth particle then is given by $f_m(k)=f(k)e^{-imak}$. The overlap and transmission matrices [Eq. (17)] are given by the Fourier transforms,

$$\mathsf{S}^{f}_{mn}=\int\frac{dk}{2\pi}|f(k)|^2 e^{i(m-n)ka},$$

$$\mathsf{T}^{f}_{mn}=\int\frac{dk}{2\pi}|f(k)|^2 T_k e^{i(m-n)ka}. \qquad (56)$$

These are Toeplitz matrices as their elements depend only on the difference $m-n$ between indices. In the limit $N\to\infty$, the determinants of the Toeplitz matrices S^{f} and $\mathsf{S}^{f}-\mathsf{T}^{f}+\mathsf{T}^{f}e^{i\lambda}$ can be evaluated by reducing the integral over k space to an integral over the first Brillouin zone $[0,2\pi/a]$ and using Szegő's theorem[20,21] (see Ref. 20 and Appendix),

$$\log\det\mathsf{S}^{f}\sim N\int_0^{2\pi}\frac{d\theta}{2\pi}\log\left\{\frac{1}{a}\sum_{m\in\mathbb{Z}}|f[(\theta+2\pi m)/a]|^2\right\},$$

$$\log\det(\mathsf{S}^{f}-\mathsf{T}^{f}+\mathsf{T}^{f}e^{i\lambda}) \sim N\int_0^{2\pi}\frac{d\theta}{2\pi}\log\left\{\frac{1}{a}\sum_{m\in\mathbb{Z}}|f[(\theta+2\pi m)/a]|^2 [1-T_{(\theta+2\pi m)/a}+T_{(\theta+2\pi m)/a}e^{i\lambda}]\right\}. \qquad (57)$$

The logarithm of these determinants scales linearly with N, a result that has to be expected as correlations between particles vanish at large separation. Combining the results [Eq. (57)] and replacing the integration over the angle θ by an integration over the first Brillouin zone $k\in[0,2\pi/a]$, we find the generating function in the form

$$\log\chi_N(\lambda)=Na\int_0^{2\pi/a}\frac{dk}{2\pi}\log(1-\tau_k+\tau_k e^{i\lambda}) \qquad (58)$$

with the effective scattering probabilities,

$$\tau_k=\frac{\sum_{m\in\mathbb{Z}}|f(k+2\pi m/a)|^2 T_{k+2\pi m/a}}{\sum_{m\in\mathbb{Z}}|f(k+2\pi m/a)|^2}, \qquad (59)$$

which denote transmission probabilities (with $0\le\tau_k\le 1$) averaged over higher harmonics $2\pi m/a$ with weight $|f(k+2\pi m/a)|^2$.

Let us apply this result to wave packets generated by Lorentzian voltage pulses. As shown in Ref. 10, a unit-flux (i.e., $c\int dtV(t)=hc/e=\Phi_0$) Lorentzian voltage pulse $eV_{t_0}(t)=2\hbar\gamma/[(t-t_0)^2+\gamma^2]$, parametrized by its width γ and time of appearance t_0, excites a single particle with wave function $f_{x_0}(k)=\sqrt{4\pi\xi}e^{-\xi k-ix_0k}\Theta(k)$ moving through the quantum wire [$\Theta(k)$ denotes the unit-step function; we remind that k is

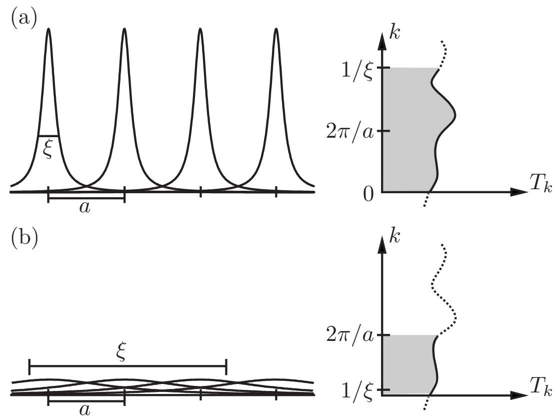

FIG. 3. (a) Train of nonoverlapping wave packets, $\xi \ll a$. Each particle is transmitted independent of the others with a transmission probability $\int (dk/2\pi)|f(k)|^2 T_k$ depending only on its momentum distribution $f(k)$. For wave packets with width ξ, these probe transmission probabilities for momenta up to $1/\xi$. (b) Train of strongly overlapping wave packets, $\xi \gg a$. If the particles were transmitted independent of each other (no exchange effects), they would probe transmission probabilities T_k in the range up to $1/\xi \approx 0$. Due to exchange effects, the particles fill up a Fermi sea determined by the density $1/a$. Therefore, the particle train probes transmission probabilities for momenta in the interval $[0, 2\pi/a]$. This (stationary) state can be seen as a wave-packet analog of the constant-voltage setup.

measured with respect to the Fermi momentum k_F]. Here e is the charge of the particle, $x_0 = v_F t_0$ parametrizes the position, and $\xi = v_F \gamma$ is the real-space width of the wave packet. A periodic sequence of unit-flux voltage pulses $V(t) = \sum_{m\in\mathbb{Z}} V_{ma/v_F}(t)$ applied to an interval to the left of the scatterer and driving one particle per time interval a/v_F generates the transmission probabilities,

$$\tau_k = (1 - e^{-4\pi\xi/a}) \sum_{m\geq 0} e^{-4\pi m\xi/a} T_{k+2\pi m/a}. \tag{60}$$

For nonoverlapping wave packets $\xi \ll a$ [cf. Fig. 3(a)], exchange effects are absent. The sum in Eq. (60) becomes an integral and the transmission probabilities assume the simple form $\tau_k = \int (dk'/2\pi)|f(k')|^2 T_{k'} = \langle \mathcal{T} \rangle$ [as easily obtained from Eq. (59) by replacing the sums with integrals], independent of k; every particle probes the transmission probabilities with its weight $|f(k)|^2$. In the opposite limit, $\xi \gg a$, i.e., for flat wave packets which are strongly overlapping, the result for distinguishable particles is $\tau_k^{\mathrm{dist}} \to T_0$. However, exchange effects force the system to fill the first Brillouin zone $k \in [0, 2\pi/a]$ and the particles probe the transmission within this energy interval $\tau_k = T_k$ [as obtained from Eq. (60) retaining only the $m=0$ term] [cf. Fig. 3(b)]. Taking the limit $\xi \to \infty$ corresponds to the case of a constant applied voltage V of magnitude $eV = 2\pi\hbar v_F/a$ (we remind that the time integral over one voltage pulse generates one flux unit $\Phi_0 = hc/e$ and pulses are separated in time by a/v_F) and the generating function assumes the form

$$\log \chi_N(\lambda) = N \frac{2\pi\hbar v_F}{eV} \int_0^{eV/\hbar v_F} \frac{dk}{2\pi} \log(1 - T_k + T_k e^{i\lambda}). \tag{61}$$

This result then is the characteristic function for the full counting statistics for a constant voltage V applied to the left of the scatterer including an energy-dependent scatterer. The equivalent result has been found in Ref. 19 if we perform the *ad hoc* replacement of the particle number N by a "measuring time" t, $N \to t v_F/a = teV/2\pi\hbar$ (note that, although this replacement appears sensible, it is nonrigorous as we have assumed the limit $t \to \infty$ within the present scattering formalism; we will further comment on this later).

The next order term in the asymptotic expansion for $N \to \infty$ can be obtained using the generalization of Szegő's theorem[20,21] [Fisher-Hartwig conjecture, cf. Eq. (A10)] and is given by

$$\Delta \log \chi_N(\lambda) = \frac{\log N}{4\pi^2} \log^2 \left[\frac{1 - T_{2\pi/a} + T_{2\pi/a} e^{i\lambda}}{1 - T_0 + T_0 e^{i\lambda}} \right]. \tag{62}$$

The logarithmic nature of this correction is due to the energy dependence of the transmission coefficient T_k, in particular, its jump $T_0 \neq T_{2\pi/a}$ across the first Brillouin zone $k \in [0, 2\pi/a]$ [for $T_0 = T_{2\pi/a}$ the correction term is of order unity; see Eq. (A7)]. The correction for the noise term is given by

$$\Delta \langle\langle n^2 \rangle\rangle = \frac{(T_{2\pi/a} - T_0)^2}{2\pi^2} \log N \tag{63}$$

and similar corrections are obtained for the third- and higher-order cumulants.

V. GENERALIZATIONS

A. Unitary evolution and time-dependent counting

We want to generalize the generating function χ_N as given by Eq. (14) to account both for the specific time evolution of the scattering state and for different counting procedures. Throughout this discussion, it is convenient to apply the Dirac notation and we rewrite Slater determinant (13) in the form

$$|\Psi\rangle = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \mathrm{sgn}(\pi) |\phi_{\pi(1)}\rangle \otimes \cdots \otimes |\phi_{\pi(N)}\rangle. \tag{64}$$

Equation (64) describes the initial N-particle wave function at time $t=0$ composed of orthonormalized one-particle states $|\phi_m\rangle$ (here, π denotes an element of the permutation group S_N). The choice of orthonormalized wave packets is only for convenience: as seen in Sec. II D, a Slater determinant is invariant under general linear combination of states; it is composed of, in particular, an orthonormalized basis that can be chosen.

Let

$$\mathcal{U} = \exp\left[-\frac{i}{\hbar}\int_0^t dt' \mathcal{H}(t')\right] \quad (65)$$

be the unitary evolution operator generated by the single-particle Hamiltonian $\mathcal{H}(t)$.[37] In the absence of interaction, the evolution of the total system is governed by the product operator $\Gamma_N(\mathcal{U})$, where, given a one-particle operator $\mathcal{O}$, we define the N-particle operator,

$$\Gamma_N(\mathcal{O}) = \underbrace{\mathcal{O} \otimes \cdots \otimes \mathcal{O}}_{N \text{ times}}, \quad (66)$$

acting simultaneously on all N particles. While we restrict ourselves to noninteracting systems, we still allow for a time-dependent scattering potential which can generate inelastic processes. The final state at time t is given by $|\Psi_{\text{out}}\rangle = \Gamma_N(\mathcal{U})|\Psi\rangle$. Including the counting field $e^{\pm i\lambda/2}$, the wave function assumes the form

$$|\Psi_{\text{out}}^{\pm}\rangle = \Gamma_N(e^{\pm i\lambda \mathcal{Q}/2})|\Psi_{\text{out}}\rangle = \Gamma_N(e^{\pm i\lambda \mathcal{Q}/2}\mathcal{U})|\Psi\rangle, \quad (67)$$

where $\mathcal{Q}$ is a projector ($\mathcal{Q}^2 = \mathcal{Q}$ and $\mathcal{Q}^\dagger = \mathcal{Q}$) on that part of the wave function that has been counted. For example, in the original setup of Ref. 4 with a spin at position x_0 and particles incoming from the left, the operator $\mathcal{Q}_t = \int_I dx |x\rangle\langle x|$ projects onto the causal interval $I = [x_0, x_0 + v_F t]$ (no such operator $\mathcal{Q}$ mimicking a spin counter can be defined for particles incident from both sides); hence that part of the wave function which passed the counter during the time t picks up an additional phase $e^{\pm i\lambda/2}$. Note that it is always the full phase λ which is picked up, as the particle is either measured (eigenvalue 1 of $\mathcal{Q}$) or not (eigenvalue 0 of $\mathcal{Q}$). The characteristic function of the full counting statistics is given by the overlap (fidelity),

$$\chi_N(\lambda) = \langle \Psi_{\text{out}}^{-}|\Psi_{\text{out}}^{+}\rangle = \langle \Psi|\Gamma_N(\mathcal{U}^\dagger e^{i\lambda\mathcal{Q}}\mathcal{U})|\Psi\rangle, \quad (68)$$

of the forward- and back-propagating wave functions measured with opposite spin states.

Next, we exploit that the expectation value of a product operator $\Gamma_N(\mathcal{O})$ in a Slater-determinant state can be written as a determinant of one-particle matrix elements $\langle \phi_m|\mathcal{O}|\phi_n\rangle$ in the Hilbert space H_N spanned by the states $|\phi_m\rangle$,

$$\begin{aligned}\langle \Psi|\Gamma_N(\mathcal{O})|\Psi\rangle &= \frac{1}{N!}\sum_{\pi,\pi' \in S_N} \text{sgn}(\pi \circ \pi')\prod_{m=1}^{N}\langle \phi_{\pi(m)}|\mathcal{O}|\phi_{\pi'(m)}\rangle \\ &= \frac{1}{N!}\sum_{\pi,\pi'' \in S_N} \text{sgn}(\pi'')\prod_{m=1}^{N}\langle \phi_m|\mathcal{O}|\phi_{\pi''(m)}\rangle \\ &= \det\langle \phi_m|\mathcal{O}|\phi_n\rangle. \end{aligned} \quad (69)$$

This formula is at the origin of (most) results which cast the characteristic function of the full counting statistics into a determinant form. Making use of Eq. (69), we can rewrite the characteristic function [Eq. (68)] as the determinant

$$\chi_N(\lambda) = \det\langle \phi_m|e^{i\lambda\mathcal{U}^\dagger\mathcal{Q}\mathcal{U}}|\phi_n\rangle = \det\langle \phi_m|1 - \mathcal{T}_Q + \mathcal{T}_Q e^{i\lambda}|\phi_n\rangle \quad (70)$$

with

$$\mathcal{T}_Q = \mathcal{U}^\dagger\mathcal{Q}\mathcal{U}. \quad (71)$$

In going from the first to the second line in Eq. (70), the exponential has been expanded and use has been made of the fact that $\mathcal{T}_Q$ is a projector. With $\mathcal{T}_Q$ as a projector in the one-particle Hilbert space H, its eigenvalues in the subspace H_N lie between 0 and 1 and Eq. (70) leads to a generalized binomial statistics.

In order to familiarize us with this new formula, we reproduce the results of Section II B. We then are interested in the situation where the initial state $|\Psi\rangle$ is localized to the left of the scattering region and the final state describes the $t \to \infty$ asymptotic behavior where all particles have completed the scattering process. Within the basis of states left or right of the scatterer with momentum k, the asymptotic form of the propagator is given by the unitary (scattering) matrix,

$$U_k^\infty = \begin{pmatrix} r_k & t_k' \\ t_k & r_k' \end{pmatrix}, \quad (72)$$

where the coefficients r_k (r_k') and t_k (t_k') are the reflection and transmission amplitudes of a particle incoming from the left (right). The total propagator assumes the form $\mathcal{U}^\infty = \int (dk/2\pi)|k\rangle_{\text{out}} U_k^\infty {}_{\text{in}}\langle k|$ where we have introduced the asymptotic states $|k\rangle_{\text{in(out)}} = (|k\rangle_{L,\text{in(out)}}, |k\rangle_{R,\text{in(out)}})$ which are in- (out-) going plane waves in the left or right lead; in a formal derivation, we have to consider the $t \to \infty$ limit of the evolution in Eq. (65) within an interaction picture with a trivial reference dynamics $\mathcal{U}_0 = \int (dk/2\pi) e^{-iv_F kt}(|k\rangle_{\text{in}}\, {}_{\text{in}}\langle k| + |k\rangle_{\text{out}}\, {}_{\text{out}}\langle k|)$. The counting operator $\mathcal{Q}$ is given by the projection on the right outgoing lead, $\mathcal{Q}_R = (0,1)^\dagger(0,1)$, and we obtain

$$\mathcal{T}_{Q_R}^\infty = \int (dk/2\pi)|k\rangle_{\text{in}}(t_k, r_k')^\dagger(t_k, r_k')_{\text{in}}\langle k|. \quad (73)$$

Since the initial single-particle wave functions ${}_{\text{in}}\langle k|\phi_m\rangle = (\langle k|\phi_m\rangle, 0)$ are located to the left of the scatterer, the characteristic function assumes the form

$$\chi(\lambda) = \det\langle \phi_m|1 - \mathcal{T} + \mathcal{T}e^{i\lambda}|\phi_n\rangle, \quad (74)$$

with $\mathcal{T} = \int (dk/2\pi) T_k |k\rangle\langle k|$, in agreement with Eq. (14); here, we have shortened the notation $|k\rangle = |k\rangle_{\text{in},L}$ in agreement with the notation above. The generalization of the result [Eq. (74)] to many channels is straightforward; the propagator [Eq. (72)] exhibits a block structure with matrices t_k and r_k describing the transmission and reflection in the channel basis, the transmission probabilities $T_k = t_k^\dagger t_k$ assume a matrix form, and the state vector $|\phi_m\rangle$ adopts an additional channel index. Assuming an implicit summation over channel indices, the form of Eq. (74) remains unchanged. The same comment holds for the spin index.

B. Density matrix: Finite temperatures

The determinant in Eq. (70) is restricted to the subspace spanned by the initial states $|\phi_m\rangle$. Introducing the projection operator $\mathcal{P} = \sum_{m=1}^{N}|\phi_m\rangle\langle \phi_m|$ onto the subspace spanned by the initial states $|\phi_m\rangle$, the determinant can be elevated to cover the whole Hilbert space. We split the total Hilbert space into the sector defined by the projector $\mathcal{P}$ and its complement projected onto $\mathcal{P}_\perp = 1-\mathcal{P}$. The operator $1-\mathcal{P}\mathcal{T}_Q+\mathcal{P}\mathcal{T}_Q e^{i\lambda}$ can be expressed in block form,

$$[1+\mathcal{P}\mathcal{T}_Q(e^{i\lambda}-1)] = \begin{bmatrix} 1+\mathcal{T}_Q(e^{i\lambda}-1) & \mathcal{T}_Q(e^{i\lambda}-1) \\ 0 & 1 \end{bmatrix}, \tag{75}$$

with the blocks operating in the $\mathcal{P}H$ and $\mathcal{P}_\perp H$ subspaces. The determinant of the upper block-diagonal matrix [Eq. (75)] is given as the product of determinant (70) in the P block and the determinant of 1 in the $P_\perp$ block and thus the generating function assumes the form

$$\chi_N(\lambda) = \det(1-\mathcal{P}\mathcal{T}_Q+\mathcal{P}\mathcal{T}_Q e^{i\lambda}), \tag{76}$$

where the determinant is taken over the entire one-particle Hilbert space H.

Interestingly, this formula can be generalized to the case when the initial state is not a single Slater determinant but an incoherent superposition of many Slater determinants with a density matrix of the form $\Gamma(\rho)/Z$ in Fock space $F=\oplus_N H_N$, $\Gamma(\mathcal{O})=\oplus_N \Gamma_N(\mathcal{O})$, $Z=\mathrm{Tr}_{F_a}\Gamma(\rho)$, and ρ is the one-particle density matrix, e.g., $\rho=e^{-\beta(\mathcal{H}-\mu)}$ for a thermal ensemble with temperature β^{-1}, chemical potential μ, and time-independent *single-particle* Hamiltonian $\mathcal{H}$; here F_a denotes the antisymmetric sector of the Fock space. Using the trace formula,[38] $\mathrm{Tr}_{F_a}[\Gamma(\mathcal{O})]=\det(1+\mathcal{O})$, where the determinant is over the one-particle Hilbert space, the characteristic function $\chi(\lambda) =\mathrm{Tr}_{F_a}[\Gamma(\rho)\Gamma(\mathcal{U}^\dagger e^{i\lambda\mathcal{Q}}\mathcal{U})]/Z$ [cf. Eq. (68)] assumes the form

$$\begin{aligned}\chi(\lambda) &= \det(1+\rho e^{i\lambda\mathcal{U}^\dagger\mathcal{Q}\mathcal{U}})/\det(1+\rho) = \det(1-\eta+\eta e^{i\lambda\mathcal{T}_Q}) \\ &= \det(1-\eta\mathcal{T}_Q+\eta\mathcal{T}_Q e^{i\lambda})\end{aligned} \tag{77}$$

with the one-particle occupation-number operator $\eta=\rho/(1+\rho)$ (and arbitrary one-particle density matrix ρ); note again that the spectrum of $\eta\mathcal{T}_Q$ resides between 0 and 1 so that Eq. (77) denotes a generalized binomial statistics.

As an example, consider the situation of two particles incident from the left, with wave functions $\phi_1(k)$ and $\phi_2(k)$, where the process of particle generation is not deterministic but involves some success probability: let p_1 (p_2) be the probability that the first (second) particle is successfully created. In order to keep the discussion simple, we assume $\phi_1(k)$ and $\phi_2(k)$ to be orthonormalized $\langle\phi_m|\phi_n\rangle=\delta_{mn}$. The initial state can be written as a density matrix $\Gamma(\rho)/Z$ with the one-particle density matrix,

$$\rho = \frac{p_1}{1-p_1}|\phi_1\rangle\langle\phi_1| + \frac{p_2}{1-p_2}|\phi_2\rangle\langle\phi_2|. \tag{78}$$

The weights in Eq. (78) are chosen to make the single-particle occupation operator $\eta=\rho/(1+\rho)$ have the form

$$\eta = p_1|\phi_1\rangle\langle\phi_1| + p_2|\phi_2\rangle\langle\phi_2|, \tag{79}$$

i.e., p_1 (p_2) are the probabilities to occupy state 1 (2). The normalized density matrix [we use the fact that $Z =\mathrm{Tr}_{F_a}\Gamma(\rho)=\det(1+\rho)=1/(1-p_1)(1-p_2)$],

$$\begin{aligned}\Gamma(\rho)/Z &= (1-p_1)(1-p_2) \oplus p_1(1-p_2)|\phi_1\rangle\langle\phi_1| \\ &\quad + p_2(1-p_1)|\phi_2\rangle\langle\phi_2| \oplus p_1p_2|\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|,\end{aligned} \tag{80}$$

consists of three terms. The first term describes the zero particle sector which occurs with probability $(1-p_1)(1-p_2)$. The second term involves one-particle states: $p_1(1-p_2)$ [$p_2(1-p_1)$] is the probability that only the first (second) particle is created. The third term shows that the probability to observe a Slater determinant of both states ϕ_1 and ϕ_2 is p_1p_2; we have omitted terms which involve tensor products of more than one projector on the same state as they have no weight on the antisymmetric part of the Hilbert space. The generating function of full counting statistics is given by Eq. (77),

$$\begin{aligned}\chi(\lambda) &= \det(1-\eta\mathcal{T}^\infty_{Q_R}+\eta\mathcal{T}^\infty_{Q_R}e^{i\lambda}) \\ &= (1-p_1T+p_1Te^{i\lambda})(1-p_2T+p_2Te^{i\lambda}),\end{aligned} \tag{81}$$

for the simplest case of asymptotic scattering with an energy-independent transmission probability, $T_k=T$.

VI. CONSTANT VOLTAGE

Many results in the literature so far have been obtained in the stationary regime where a constant voltage V is applied across the wire for long measuring times $teV/\hbar \gg 1$.[1,39–41] Here, we discuss a wave-packet analog of the constant-voltage case. Contrary to the discussion in Sec. IV involving a nonstationary finite train of N particles with the spin counter measuring all the time $t\to\infty$, here, we consider a stationary situation in the thermodynamic limit ($N,L\to\infty$ with fixed density $n=N/L$, where L is the system size) with two reservoirs disbalanced by the applied voltage V and the counting extending over a finite time t.

We start with N particles residing in (left incident) scattering states,

$$\varphi_k(x) = (e^{ikx}+r_ke^{-ikx})\Theta(-x)+t_ke^{ikx}\Theta(x), \tag{82}$$

with energies $\hbar\varepsilon=\hbar v_F k$ between E_F and E_F+eV. The scatterer is positioned at the origin. In order to regularize the problem, we go over to wave packets $\phi_m(x)$: we split the momentum interval $[0,eV/v_F]$ into compartments of width $\hbar\varkappa=eV/v_FN$ and define the weights

$$f_m(k) = \begin{cases} \sqrt{2\pi/\varkappa} & \text{if } \varkappa(m-1)\le k\le\varkappa m \\ 0 & \text{elsewhere,}\end{cases} \tag{83}$$

with $m\in\{1,\ldots,N\}$. With the real weights $f_m(k)$, the (normalized) wave packets

$$\phi_m(x) = \int\frac{dk}{2\pi}f_m(k)\varphi_k(x) \tag{84}$$

define states centered around the origin. Note that adding arbitrary global phases to the wave packets $\phi_m(x)$ does not change their Slater determinant [up to a trivial global phase of the many-body wave function, see Eq. (64)]. Keeping V constant and letting $\varkappa\to 0$, the wave packets spread out in real space, the particle number N goes to infinity, the homogeneous particle density assumes the finite value $eV/2\pi\hbar v_F$, and the resulting current $\langle\mathcal{I}\rangle=(e/h)TV$ is constant in time [cf. Eq. (101)]. This procedure then properly emulates the constant-voltage setup, as it generates the identical zero-temperature density matrix as the one obtained in a second quantization formulation by filling scattering states within the interval of width eV.

In making use of expression (70), we need the time evolution of the wave packets,

$$\phi_m(x;t) = \int \frac{dk}{2\pi} e^{-iv_F kt} f_m(k)\varphi_k(x), \tag{85}$$

as well as the counting operator $\mathcal{Q}_t=\int_I dx|x\rangle\langle x|$ projecting particles on the space interval $I=[x_0,x_0+v_Ft]$, where we assume the counter to be placed to the right of the origin, $x_0>0$.

A. Generalized binomial statistics

The characteristic function $\chi_t(\lambda)$ is the determinant of the matrix [cf. Eq. (70)],

$$\begin{aligned}\langle\phi_m|e^{i\lambda\mathcal{U}^\dagger\mathcal{Q}_t\mathcal{U}}|\phi_n\rangle &= \langle\phi_m(t)|e^{i\lambda\mathcal{Q}_t}|\phi_n(t)\rangle \\ &= \int dx\,\phi_m(x;t)^*\langle x|e^{i\lambda\mathcal{Q}_t}|x\rangle\phi_n(x;t) \\ &= \delta_{mn} + (e^{i\lambda}-1)\mathbf{Q}_{mn},\end{aligned} \tag{86}$$

$$\chi_t(\lambda) = \det[\delta_{mn} + (e^{i\lambda}-1)\mathbf{Q}_{mn}] \tag{87}$$

with

$$\mathbf{Q}_{mn} = \int \frac{dk'dk}{4\pi^2} t_k f_n(k) K_t(k-k') t^*_{k'} f^*_m(k') \tag{88}$$

and the kernel

$$\begin{aligned}K_t(q) &= \int_{x_0}^{x_0+v_Ft} dx e^{iq(x-v_Ft)} = \frac{e^{iqx_0}(1-e^{-iqv_Ft})}{iq} \\ &= 2e^{iq(x_0-v_Ft/2)}\frac{\sin(qv_Ft/2)}{q}.\end{aligned} \tag{89}$$

The matrix $\mathbf{Q}_{mn}$ is a Hermitian matrix with real eigenvalues $0\le\tau_m(t)\le 1$; hence the associated full counting statistics is generalized binomial for all times. The same result can be retrieved from Ref. 34 with an appropriate choice for the time-dependent scatterer. In the following, we discuss various limits for the generating function $\chi_t(\lambda)$.

B. Short measuring time

Assuming that N is large enough so that t_k does not change appreciably over the interval $\varkappa$, i.e., $\varkappa\partial_k t_k \ll 1$, the amplitude t_k can be taken out of the integral in Eq. (88). Assuming furthermore that the measurement time t is short, $|q|v_Ft \le teV/\hbar \ll 1$, we can expand $K_t(q)$ and obtain (to lowest order in $teV/\hbar$)

$$\begin{aligned}\chi_t^{\ll}(\lambda) = \det\Bigg[&\delta_{mn} + (e^{i\lambda}-1)t^*_{\varkappa m}t_{\varkappa n}\varkappa v_F t \\ &\times e^{i\varkappa x_0(n-m)}\frac{2\sin^2(\varkappa x_0/2)}{\pi\varkappa^2 x_0^2}\Bigg].\end{aligned} \tag{90}$$

The second term involves a matrix product $(v_1,v_2,\ldots,v_N)^\dagger(v_1,v_2,\ldots,v_N)$ of a vector and its dual, where $v_m=t_{\varkappa m}e^{i\varkappa x_0 m}$, and hence can be written as a projector, in Dirac notation, $\mu|v\rangle\langle v|$ with $\mu=2(e^{i\lambda}-1)\varkappa v_F t\sin^2(\varkappa x_0/2)/\pi\varkappa^2 x_0^2$. The determinant $\det(1+\mu|v\rangle\langle v|)$ then is given by the product of eigenvalues $1+\mu\langle v|v\rangle$ (in the direction of $|v\rangle$) and 1 (in the complement), $\det(1+\mu|v\rangle\langle v|)=1+\mu\langle v|v\rangle$, and we obtain

$$\begin{aligned}\chi_t^{\ll}(\lambda) &= 1 + (e^{i\lambda}-1)\varkappa v_F t\frac{2\sin^2(\varkappa x_0/2)}{\pi\varkappa^2 x_0^2}\sum_{m=1}^{N} T_{\varkappa m} \\ &\overset{(\varkappa\to 0)}{\to} 1 + \alpha(e^{i\lambda}-1),\end{aligned} \tag{91}$$

with (note that $N\varkappa=eV/\hbar v_F$)

$$\alpha = tv_F\int_0^{eV/\hbar v_F}\frac{dk}{2\pi}T_k \le teV/2\pi\hbar \ll 1 \tag{92}$$

(note that the dependence on the counter position x_0 has disappeared from the parameter $\alpha\propto t$ in the constant-voltage limit $\varkappa\to 0$). Pushing the calculation to higher order in $teV/\hbar$, we find that in the expansion of $\chi_t^{\ll}(\lambda)$ both, second- and third-order terms, vanish and the next correction appears only in fourth order,

$$\begin{aligned}\Delta\chi_t^{\ll}(\lambda) &= \frac{(e^{i\lambda}-1)^2(tv_F)^4}{24}\int_0^{eV/\hbar v_F}\frac{dkdk'}{(2\pi)^2}T_kT_{k'}(k-k')^2 \\ &\le \left(\frac{e^{i\lambda}-1}{24\pi}\right)^2\left(\frac{teV}{\hbar}\right)^4.\end{aligned} \tag{93}$$

Hence, for short measuring times the majority of counts involve either no or a single particle, while the observation of two-particle events $P_2\le(teV/\hbar)^4/(24\pi)^2$ is strongly suppressed, a consequence of the Pauli exclusion principle. Note that in the short measuring time limit, the specific nature of the counting device matters. Above, we have assumed that all intrinsic time scales of the counter are much shorter than the measuring time. Furthermore, we have neglected the effect of the Fermi sea which will produce an additional contribution. Nevertheless, even modeling the counter more realistically, the Pauli principle with its reduction in two- and more-particle events is expected to reveal itself. Furthermore, it is possible to realize an experiment where the effect of the additional Fermi sea is absent: by applying a voltage to a quantum wire which is larger than the Fermi energy, particles incident from the right are blocked by the band bottom and only left going states within an energy interval E_F (replacing the bias eV) contribute to the particle current.[42]

Note that the generalized binomial statistics [Eq. (3)] reduces to the simple Poissonian result,

$$\log \chi(\lambda) = \sum_m \tau_m (e^{i\lambda} - 1), \tag{94}$$

in the limit of small generalized transmission probabilities $\tau_m \ll 1$; the result then only depends on one parameter $\Sigma_m \tau_m = \mathrm{tr}\, \mathcal{T}_{Q_t}$. In the limit of short measuring times, the smallness of the transmission eigenvalues is imposed by the small space interval in the projection $\mathcal{Q}_t$ and $\Sigma_m \tau_m = \alpha$ [see Eq. (92)]. The same result is obtained in the long-time limit [see Eq. (98)], provided the transmission probabilities T_k themselves are small.

C. Large measuring times

In the asymptotic limit of $t \to \infty$, the kernel $K_t(q)$ ensures energy or momentum conservation, rendering the problem diagonal in the momentum basis.[1,19] However, adopting the $t \to \infty$ asymptotic limit is incompatible with a regular derivation of a finite result. Here, we consider instead the case of large but finite measuring time t while adopting the limit of infinite particle number $N \to \infty$ when letting the width $\hbar\varkappa = eV/v_F N$ go to zero at constant voltage V.

In the limit $N \to \infty$, the characteristic function $\chi_t(\lambda)$, which is the determinant of the matrix in Eq. (86) [cf. Eq. (70)], is given by

$$\chi_t(\lambda) = \det(1 - \mathcal{P}\mathcal{T}\mathcal{Q}_t + \mathcal{P}\mathcal{T}\mathcal{Q}_t e^{i\lambda}), \tag{95}$$

where $\mathcal{P} = \int_0^{eV/\hbar v_F} (dk/2\pi) |k\rangle\langle k|$ is the projector on the subspace of occupied states; form (95) can be obtained from Eq. (86) introducing the projector $\mathcal{P}$ to extend the determinant over the whole Hilbert space [cf. Eqs. (75) and (76)] and using the determinant identity $\det(1+\mathcal{A}\mathcal{B}) = \det(1+\mathcal{B}\mathcal{A})$ to shuffle $t_k'^*$ to the left of K_t, which itself is the momentum representation of the projector $\mathcal{Q}_t$. The expression [Eq. (95)] then corresponds to Eq. (76) with the substitution $\mathcal{T}_Q = \mathcal{T}\mathcal{Q}_t$. As $\mathcal{Q}_t$ is a projector, $\mathcal{Q}_t^2 = \mathcal{Q}_t$, we can rewrite Eq. (95) as $\det[1 + (e^{i\lambda} - 1)\mathcal{Q}_t \mathcal{P}\mathcal{T}\mathcal{Q}_t]$. This determinant only needs to be calculated in the subspace $\mathcal{Q}_t H$ as the matrix is unity in the complement. In the subspace $\mathcal{Q}_t H$, we use the orthonormal real-space [rather than k space; see Eq. (83)] basis,

$$g_l(x) = \begin{cases} 1/\sqrt{\epsilon} & \text{if } \epsilon(l-1) \le x - x_0 \le \epsilon l \\ 0 & \text{elsewhere,} \end{cases} \tag{96}$$

with $\epsilon = t v_F / L$ as the width of a real-space segment and $l \in \{1, \ldots, L\}$. The matrix elements of $\mathcal{P}\mathcal{T}$ assume the form

$$\langle g_l | \mathcal{P}\mathcal{T} | g_m \rangle = \int_0^{eV/\hbar v_F} \frac{dk}{2\pi} T_k \langle g_l | k \rangle \langle k | g_m \rangle = \int_0^{eV/\hbar v_F} \frac{dk}{2\pi} T_k \frac{4 \sin^2(\epsilon k/2)}{\epsilon k^2} e^{i(l-m)k\epsilon}, \tag{97}$$

i.e., they form a Toeplitz matrix. Applying Szegő's theorem[20,21] and taking the limit of large t and L with $\epsilon = v_F t / L$ fixed but small [$\epsilon \ll \hbar v_F / eV$, hence $4 \sin^2(\epsilon k/2)/\epsilon k^2 \approx \epsilon$] we obtain the generating function

$$\log \chi_t(\lambda) = t v_F \int_0^{eV/\hbar v_F} \frac{dk}{2\pi} \log(1 - T_k + T_k e^{i\lambda}) \tag{98}$$

(cf. Appendix).

Using the generalization of Szegő's theorem[20,21] [Fisher-Hartwig conjecture; see Eqs. (A7) and (A10)],[43] it is possible to calculate the next order term. As the argument of the logarithm (cf. Eq. (A10), note that $x(\theta)$ is to be replaced by $\Sigma_{n\in\mathbb{Z}} P_{(\theta+2\pi n)/\epsilon}[1 + (e^{i\lambda} - 1) T_{(\theta+2\pi n)/\epsilon}]$ with $P_k = 1$, $k \in [0, eV/\hbar v_F]$, and $P_k = 0$ otherwise) exhibits discontinuities at $k=0$ and $k = eV/\hbar v_F$, the correction to the leading term is given by the two contributions originating from the jumps at $k = 0, eV/\hbar v_F$,

$$\Delta \log \chi_t(\lambda) = \frac{\log(t/t_0)}{4\pi^2} \sum_{k=0,eV/\hbar v_F} \log^2[1 + (e^{i\lambda} - 1) T_k], \tag{99}$$

with t_0 as some small time cutoff; this result leads to logarithmic corrections for the second-order and all higher cumulants. For the noise, the correction is given by[19,27]

$$\Delta\langle\langle n^2 \rangle\rangle = \frac{T_0^2 + T_{eV/\hbar v_F}^2}{2\pi^2} \log(t/t_0). \tag{100}$$

Here, the logarithmic corrections in Eqs. (99) and (100) are due to fluctuations in the number of particles in a finite interval of length $v_F t$. Therefore, fluctuations do not disappear for $T=1$ as in Eqs. (62) and (63) where the number of particles is fixed and noise stems only from partitioning. In addition to the noise originating from the voltage bias, there is an equilibrium contribution due to the Fermi sea at any finite measuring time (cf. Sec. VI B). For asymptotically large times, the first contribution grows logarithmically in time.[2]

The reason why Szegő's theorem[20,21] is applicable to the matrix [Eq. (97)] is the presence of time translation invariance: matrix elements between states localized at two different places or times depend only on their space or time separation and hence they form a Toeplitz matrix. The same reasoning does not apply to the momentum basis and that is why we could not apply Szegő's theorem[20,21] directly to the matrix in Eq. (86). The result [Eq. (98)] [and its generalization to finite temperatures; cf. Eq. (113)] has been found by Schönhammer[19] using a double projection in his counting procedure: instead of relying on Szegő's theorem,[20,21] use has been made of the relation $\log \det(1+M) = \mathrm{tr} \log(1+M)$, followed by an expansion of the logarithm. Evaluating the trace of each term, the phase factors [see Eq. (89)] appearing in the cyclic product of the kernel K_t cancel mutually. In the long-time limit, the kernels become diagonal (ensuring energy conservation) with one of them contributing a factor of t, thus rendering the cumulant generating function $\log \chi_t(\lambda)$ linear in t. In alternative approaches, use has been made of a mapping onto the Riemann-Hilbert problem[27] (this procedure enables the calculation of the leading term as well as the logarithmic correction [Eq. (99)]) or of time periodicity[2,8] introduced in order to render $\log \chi_t$ extensive in t (this way, only the term linear in t is obtained).

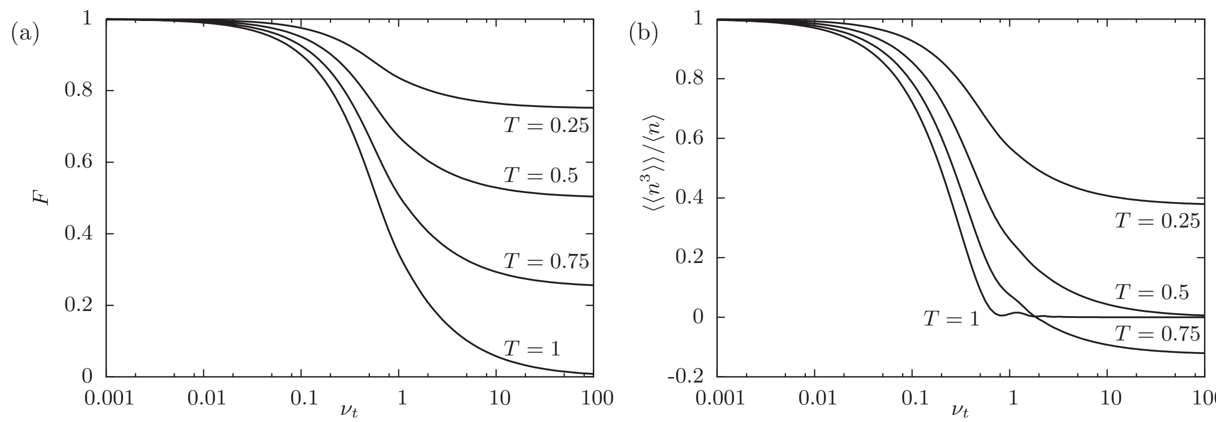

FIG. 4. Fano factor $F=\langle\langle n^2\rangle\rangle/\langle n\rangle$ in (a) and third moment $\langle\langle n^3\rangle\rangle/\langle n\rangle$ in (b) for constant voltage with energy-independent transmission probabilities $T=0.25,0.5,0.75,1$, as a function of the incident-particle number $\nu_t=teV/2\pi\hbar$. Note that the Fano factor approaches the binomial value $F=1-T$ for $\nu_t\gg 1$ whereas for $\nu_t\ll 1$ it is always close to 1 irrespective of the transmission probability T. The third cumulant interpolates as a function of ν_t between the Poissonian value $\langle\langle n^3\rangle\rangle/\langle n\rangle=1$ [cf. Eq. (94)] and the binomial result $\langle\langle n^3\rangle\rangle/\langle n\rangle=T(1-T)(1-2T)$ [cf. Eq. (98)]. The oscillations (especially around $\nu_t\approx 1$ for $T=1$) are due to the sharp edge in the occupation number at $k=eV/\hbar v_F$.

D. Fano factor for intermediate regime

In order to understand the crossover between the short- and long-time behaviors of the carrier distribution, we calculate the Fano factor F and present the result as a function of $\nu_t=teV/2\pi\hbar$ (the incident-particle number during time t) in Fig. 4(a) for several values of the transmission coefficient T (for a scatterer with energy-independent transmission). For small times, the distribution is Poissonian and hence $F(\nu_t\to 0)\to 1$. The binomial distribution valid at large times provides the asymptotics $F(\nu_t\to\infty)\to(1-T)$. In order to find the crossover in between, we determine the matrix $\mathbf{Q}$ [see Eq. (88)],

$$\mathbf{Q}_{mn}\overset{(\varkappa\to 0)}{\to} t^*_{\varkappa m}t_{\varkappa n}e^{i(n-m)\varkappa(x_0-v_Ft)}\frac{\sin[(n-m)\varkappa v_Ft/2]}{\pi(n-m)},$$

in terms of which the characteristic function assumes simple form (87) and hence $\log\chi_t(\lambda)=\mathrm{tr}\log[\delta_{mn}+(e^{i\lambda}-1)\mathbf{Q}_{mn}]$ (again, we consider the limit $\varkappa\to 0$ at fixed voltage V). The average transmitted charge $\langle n\rangle=-i\partial_\lambda\log\chi_t|_{\lambda=0}$,

$$\langle n\rangle=\mathrm{tr}\,\mathbf{Q}=tv_F\int_0^{eV/\hbar v_F}\frac{dk}{2\pi}T_k, \tag{101}$$

grows linearly with the measuring time t; the above result coincides with those obtained from the short and long-time expressions (91) and (98). The noise $\langle\langle n^2\rangle\rangle=-\partial_\lambda^2\log\chi_t|_{\lambda=0}$ assumes the form

$$\begin{aligned}\langle\langle n^2\rangle\rangle&=\mathrm{tr}\,\mathbf{Q}-\mathrm{tr}\,\mathbf{Q}^2\\&=\langle n\rangle-\int_0^{eV/\hbar v_F}\frac{dk'dk}{\pi^2}T_{k'}T_k\frac{\sin^2[(k'-k)v_Ft/2]}{(k'-k)^2}\end{aligned} \tag{102}$$

(in the limit $\varkappa\to 0$ considered here, both momenta do not depend on the position x_0 of the counter, as the wave packets are infinitely spread). In order to keep the analysis simple, we assume an energy-independent transmission probability, $T_k=T$, over the interval $[0,eV/\hbar v_F]$. The average charge then is given by

$$\langle n\rangle=TteV/2\pi\hbar=T\nu_t. \tag{103}$$

The Fano factor $F=\langle\langle n^2\rangle\rangle/\langle n\rangle$ can be cast into the form

$$F=1-Tf(\nu_t) \tag{104}$$

with

$$f(\nu_t)=\int_{-1}^{1}dx(1-|x|)\frac{\sin^2(\pi\nu_t x)}{\pi^2\nu_t x^2}. \tag{105}$$

For small times $\nu_t\ll 1$,

$$f(\nu_t)=\nu_t-\frac{\pi^2}{18}\nu_t^3+\mathcal{O}(\nu_t^5), \tag{106}$$

while f approaches unity in the long-time limit $\nu_t\gg 1$ ($\gamma\approx 0.5772$ is Euler's constant),

$$f(\nu_t)=1-\frac{\log(2\pi\nu_t)+1+\gamma}{\pi^2\nu_t}+\mathcal{O}(\nu_t^{-3}). \tag{107}$$

The corrections to the simple binomial result produce a logarithmic in time increase in the noise $\langle\langle n^2\rangle\rangle$; result (107) coincides with Eq. (100) for the case of energy-independent scattering probabilities $T_k=T$. This logarithmic dependence in the noise is due to the fluctuations in the number of electrons in a finite segment of the wire.[1] Analogously, the third cumulant $\langle\langle n^3\rangle\rangle$ can be calculated; the (numerical) results, shown in Fig. 4(b), interpolate between the Poissonian value $\langle\langle n^3\rangle\rangle/\langle n\rangle=1$ for short times and the binomial result $\langle\langle n^3\rangle\rangle/\langle n\rangle=T(1-T)(1-2T)$ for long measuring times.

E. Finite temperature

We consider the case where particles are emitted from a lead at finite temperature into vacuum, i.e., we assume a single Fermi reservoir of particles (incident from the left) which are scattered with energy-dependent transmission probabilities (to the right). At finite temperature, scattering states are occupied according to the Fermi-Dirac occupation as described by the one-particle operator $\eta=[e^{\beta(\mathcal{H}-\mu)}+1]^{-1}$. The characteristic function $\chi_t(\lambda)$ is given by Eq. (77), with $\mathcal{T}_Q=\mathcal{T}\mathcal{Q}_t$ and the interval $I=[x_0,x_0+v_F t]$ defining the projector $\mathcal{Q}_t$ [cf. Sec. VI C]. Following essentially the calculation in Sec. VI C, i.e., calculating the determinant in basis (96) and applying Szegő's theorem,[20,21] the result

$$\log \chi_t(\lambda) = t v_F \int \frac{dk}{2\pi} \log[1 + T_k n_L(k)(e^{i\lambda} - 1)], \quad (108)$$

with $n_L(k)=\langle k|\eta|k\rangle=[e^{\beta(\hbar v_F k-\mu)}+1]^{-1}$ can be obtained for fixed but long measurement times t. For high temperatures at constant particle density [$\beta\to 0$, $n_L(k)\approx e^{-\beta(\hbar v_F k-\mu)}$], all transmission eigenvalues $\tau_k=T_k/(e^{\beta(\hbar v_F k-\mu)}+1) \approx T_k e^{-\beta(\hbar v_F k-\mu)}$ approach zero. The logarithm in Eq. (108) can be expanded and the emission statistics for electrons leaving a Fermi reservoir in the high-temperature regime is given by a Poissonian statistics,[44]

$$\log \chi_t(\lambda) = (e^{i\lambda} - 1) t v_F \int \frac{dk}{2\pi} T_k e^{-\beta(\hbar v_F k-\mu)}. \quad (109)$$

The Fano factor assumes the value $F=1$, independent of T_k.

To complete the analysis, we discuss the extension of the constant-voltage result with two reservoirs to finite temperatures. We model the setup by two Fermi reservoirs with occupation numbers $n_{L/R}(k)=[\exp[\beta(\hbar v_F k-\mu_{L/R})]+1]^{-1}$ for particles incoming from the left (L) or right (R), respectively. The voltage enters via the bias of the chemical potentials $eV=\mu_L-\mu_R$.

Unfortunately, it is not possible to define a projection operator $\mathcal{Q}$ which acts *after* the evolution and which can emulate the action of the spin counter [cf. the discussion below Eq. (67)]. The reason is that there is no way to tell for a particle outgoing to the left at time $t\to\infty$ whether it was coming in from the left and was reflected at the scatterer (hence, no counting is done with the counter to the right of the scatterer) or whether it was coming in from the right and has been transmitted through the scatterer (hence passing the spin counter once). One solution to this problem is to perform a first projective measurement at the initial time;[27,29] this corresponds to replacing Eq. (77) by the expression

$$\chi_t(\lambda) = \frac{\det(1 + \rho \mathcal{U}^\dagger e^{i\lambda \mathcal{Q}_t} \mathcal{U} e^{-i\lambda \mathcal{Q}_t})}{\det(1 + \rho)}, \quad (110)$$

with the occupation operator $\eta=\rho/(1+\rho) =\int (dk/2\pi)|k\rangle_{\rm in}\, \mathrm{diag}[n_L(k),n_R(k)]_{\rm in}\langle k|$, the single-particle evolution $\mathcal{U}$ involving the scatterer but not the spin counter [cf. Eq. (72)], and $\mathcal{Q}_t$ a projector emulating the counting measurement of transmitted particles via projection of the wave functions onto the lead to the right of the scatterer. The additional factor $\exp(-i\lambda\mathcal{Q}_t)$, as compared to Eq. (77), corresponds to the additional measurement before the evolution.

In this paper, we want to stick with the spin counter as a measurement apparatus. Contrary to the situation in Sec. V, the action of the spin counter cannot be modeled by a projection onto the outgoing states, i.e., the operators $\mathcal{U}_\pm$ do not separate anymore into factors describing the scatterer and the

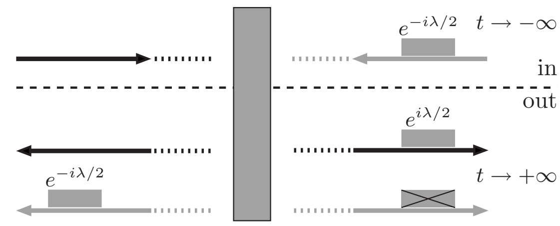

FIG. 5. Sketch of incoming ($t\to-\infty$, above the dashed line) and scattered (large gray box) outgoing ($t\to+\infty$, below the dashed line) states measured by a spin counter (small gray boxes) placed to the right of the scatterer. Left-incoming and scattered states are described by black arrows; right-incoming and scattered states correspond to gray arrows. The action of the spin counter (in the up state) is included in the expression $\mathcal{U}_+$ [see Eq. (111)]; the evolution $\mathcal{U}_+$ involves two projections, one described by $\exp(-i\lambda\mathcal{Q}_t/2)$ at time $t\to-\infty$ and the second $\exp(+i\lambda\mathcal{Q}_t/2)$ at time $t\to\infty$. The right (gray arrow) incoming state at $t\to-\infty$ acquires a phase factor $\exp(-i\lambda/2)$ (gray box on the right) in the first projection; this provides the correct counting field for its transmitted part (gray box on the left). The phase factor of the reflected part is canceled by the second counting operator (crossed box on the right). The left incoming state is unaffected by the first counting and its transmitted part acquires a phase $\exp(i\lambda/2)$ at time $t\to\infty$.

counter, $\mathcal{U}_\pm \neq e^{\pm i\lambda\mathcal{Q}_t/2}\mathcal{U}$. Therefore, we have to make use of the full evolution operators $\mathcal{U}_\pm$, $|\Psi^\pm_{\rm out}\rangle=\Gamma(\mathcal{U}_\pm)|\Psi\rangle$, in the presence of both the scatterer and the spin counter, where the index "±" refers to the two spin states of the counter. The overall evolution (cf. Fig. 5) then can be written as

$$\mathcal{U}_\pm = e^{\pm i\lambda\mathcal{Q}_t/2}\mathcal{U}e^{\mp i\lambda\mathcal{Q}_t/2}, \quad (111)$$

where $\mathcal{U}$ [cf. Eq. (72)] is the evolution without accounting for the presence of the spin counter. The generating function for the full counting statistics assumes the form

$$\chi_t(\lambda) = \frac{\det(1 + \rho e^{-i\lambda\mathcal{Q}_t/2}\mathcal{U}^\dagger e^{i\lambda\mathcal{Q}_t}\mathcal{U}e^{-i\lambda\mathcal{Q}_t/2})}{\det(1 + \rho)}. \quad (112)$$

The two counting procedures [Eqs. (110) and (112)] agree if the particles are only incident from the left, as the additional counting factors, compared to Eq. (77), contribute unity. For particles incoming from both left and right the two counting procedures do not necessarily coincide; only if $\mathcal{Q}_t$ commutes with ρ, we can shift the factor $e^{-i\lambda\mathcal{Q}_t/2}$ to the left of ρ and then cyclically permute the factors in the second term of the determinant to assert the equivalence of Eqs. (110) and (112).

The interpretation of Eq. (112) as the generating function for the full counting statistics using the spin counting procedure faces problems since Eq. (111) is not necessarily 2π periodic and hence the counting may involve a noninteger number of particles. In certain situations, however, the spin counter nevertheless leads to sensible results. In particular, for asymptotically long measuring times $t\to\infty$, the counting projection operator $\mathcal{Q}_t$ becomes basically diagonal in the energy or momentum basis. Commuting $\exp(-i\lambda\mathcal{Q}_t/2)$ with ρ and repeating the calculation in Sec. VI C, i.e., calculating the determinant in basis (96) and applying Szegő's theorem,[20,21] the result

$$\log \chi_t^{\gtrless}(\lambda) = t v_F \int \frac{dk}{2\pi} \log[1 + T_k\{n_L(k)[1-n_R(k)](e^{i\lambda}-1) + n_R(k)[1-n_L(k)](e^{-i\lambda}-1)\}], \tag{113}$$

is obtained; alternatively, result (113) can be obtained by expanding the determinant using the relation $\log\det(1+M) = \Sigma_{k=1}^{\infty}(-1)^k \operatorname{tr} M^k/k$ and determining the leading contribution in each order.[19] Away from the asymptotic limit (including also the calculation of next-to-leading-order corrections) the above commutation cannot be carried out and half-integer charges might show up.[28] A thorough discussion of the equivalence of counting procedures (110) and (112) for finite measuring times is still lacking.

VII. CONCLUSION

We have used the first-quantized wave-packet formalism to calculate the generating function $\chi_N(\lambda)$ of full counting statistics of fermionic particles in various physical situations, such as N particles incident in Slater-determinant states of rank 1 (nonentangled), rank 2 (entangled), or incoherent superpositions of Slater determinants in Fock space with undetermined particle number. Our formalism captures various features such as energy-dependent scattering probabilities as well as time-dependent scattering and time-dependent counting.

We have presented our results in determinantal form, with further simplifications explicitly unveiling a generalized binomial statistics in various cases. Applications of our results include a classification of possible statistical behavior of two-particle scattering events and a particularly simple singlet-triplet and entanglement detector. In the context of coherent transport of noninteracting (degenerate) fermions, the natural reference point in the discussion of statistical properties is the binomial distribution; energy-dependent scattering naturally shifts the noise into the sub-binomial (or generalized binomial) regime, whereas additional correlations through entanglement can generate superbinomial noise statistics.

Our results, calculated at zero temperature, remain valid for $\beta^{-1} \ll \hbar v_F/\xi$, i.e., sufficiently narrow wave packets with a small width ξ in real space. Furthermore, we have calculated the generating function for the constant-voltage case in the long-time limit for any temperature. For short measuring times our results are valid in the temperature regime $\beta^{-1} \ll eV$ and we have found a strong suppression of $P_{n\geq 2}$ due to Pauli blocking.

The central element underlying the appearance of a (sub-)binomial statistics in fermionic systems is the absence of interparticle interactions and entanglement. This result remains valid for a time-dependent scattering potential and finite temperature. We have analyzed the modification introduced by entanglement and have found that superbinomial statistics may be generated. The inclusion of interaction, particularly within the scatterer where interacting particles become entangled, remains an interesting open problem.

ACKNOWLEDGMENTS

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APPENDIX: (STRONG) SZEGŐ THEOREM (REFS. 20 and 21)

The (strong) Szegő theorem[20,21] applies to Toeplitz matrices and reduces the calculation of the asymptotic behavior of their determinants to a simple integration (plus summation) problem. We define a Toeplitz matrix starting from a complex-valued periodic function $a(\theta)$ with $a(\theta+2\pi)=a(\theta)$. In addition, we require that its winding number with respect to the origin is equal to zero. We define the Fourier coefficients

$$a_m = \int_0^{2\pi} \frac{d\theta}{2\pi} a(\theta) e^{-im\theta} \tag{A1}$$

and the associated $N\times N$ Toeplitz matrix with elements

$$[\mathbf{A}_N(a)]_{m,n} = a_{m-n} \tag{A2}$$

depending only on the difference between the indices m and n (banded matrix), $m,n=1,\ldots,N$. The strong form of the Szegő theorem[21,45] states that

$$\log\det \mathbf{A}_N(a) \sim N[\log a]_0 + \sum_{n=1}^{\infty} n[\log a]_n[\log a]_{-n} \tag{A3}$$

asymptotically for $N\to\infty$, with

$$[\log a]_n = \int_0^{2\pi} \frac{d\theta}{2\pi} \log[a(\theta)] e^{-in\theta} \tag{A4}$$

As the Fourier coefficients of $\log[a(\theta)]$. The first term in Eq. (A3) scaling with N^1 is the result of Szegő's theorem,[20,21] while its strong form applies once the sum in the second term converges—this correction then scales with N^0.

Given the Toeplitz matrix $\mathbf{X}^f = \mathbf{S}^f + (e^{i\lambda}-1)\mathbf{T}^f$ [cf. Eq. (56)], we show how to find its determinant [Eq. (57)] in the asymptotic limit of large N. Specifying the matrix elements,

$$x_{m-n} = \int \frac{dk}{2\pi} |f(k)|^2 (1 - T_k + T_k e^{i\lambda}) e^{i(m-n)ka}, \tag{A5}$$

we find the original periodic function $x(\theta)$ by calculating the Fourier series,

$$x(\theta) = \sum_{m\in\mathbb{Z}} x_m e^{im\theta} = \frac{1}{a}\sum_{m\in\mathbb{Z}} |f[(\theta+2\pi m)/a]|^2 \times [1 - T_{(\theta+2\pi m)/a} + T_{(\theta+2\pi m)/a} e^{i\lambda}]. \tag{A6}$$

Note that, while the original function $x(k)=|f(k)|^2(1-T_k + T_k e^{i\lambda})$ was defined on the real axis, the new expression $\kappa(k)=x(\theta=ak)$ is restricted to the first Brillouin zone $k \in [0,2\pi/a]$. Fourier transforming the logarithm of $x(\theta)$ according to Eq. (A4), we obtain the asymptotic expression for the determinant,

$$\log \det \mathbf{X}_N^f = N\int_0^{2\pi} \frac{d\theta}{2\pi}\log[x(\theta)] + \sum_{n=1}^{\infty} n[\log x]_n[\log x]_{-n} + o(1), \quad \text{(A7)}$$

consisting of a main term $\propto N$, a first correction staying constant as $N\to\infty$, and a remaining correction $o(1)$ vanishing as $N\to\infty$. The (logarithm of the) determinant $\mathbf{S}^f$ in Eq. (57) is derived by setting $T\equiv 0$ in Eq. (A6). Finally, we obtain the (log of the) characteristic function by simple subtraction (we replace the angle θ on the unit circle $[0,2\pi]$ by $k=\theta/a$ in the first Brillouin zone $[0,2\pi/a]$) to leading order in N,

$$\log \chi_N(\lambda) = Na\int_0^{2\pi/a} \frac{dk}{2\pi}\log(1-\tau_k+\tau_k e^{i\lambda}), \quad \text{(A8)}$$

with the effective scattering probabilities

$$\tau_k = \frac{\sum_{m\in\mathbb{Z}} |f(k+2\pi m/a)|^2 T_{k+2\pi m/a}}{\sum_{m\in\mathbb{Z}} |f(k+2\pi m/a)|^2}. \quad \text{(A9)}$$

For a function $x(\theta)$ which is continuous on the unit circle, i.e., $x(2\pi)=x(0)$, the sum in Eq. (A7) converges and the corrections to Eq. (A8) are constant when $N\to\infty$ (and similar for $s(\theta)=\Sigma_{m\in\mathbb{Z}}|f[(\theta+2\pi m)/a]|^2/a$ in the calculation of $\log\det \mathbf{S}_N^f$). A more subtle situation appears in the situation where $x(\theta)$ and/or $s(\theta)$ are discontinuous across the Brillouin zone (Fisher-Hartwig conjecture).[43] This situation is the usual case as the wave function $f(k)$ is discontinuous at the Fermi level $k=0$. Thus, the sum in Eq. (A7) is divergent and the next term in the expansion of Eq. (A8) scales with $\log N$ [cf. also Eq. (99)],

$$\Delta \log \chi_N(\lambda) = \frac{\log^2[x(0^+)/x(0^-)] - \log^2[s(0^+)/s(0^-)]}{4\pi^2}\log N, \quad \text{(A10)}$$

and is followed by a constant term. Here, the number of particles N is fixed and noise is due to partitioning; hence, the logarithmic corrections [Eq. (A10)] have to be attributed to partitioning (and not to fluctuations in the number of particles as for the constant-voltage result [Eq. (99)]). This is also consistent with the vanishing of correction (A10) for $T=1$ where $x(\theta)=e^{i\lambda}s(\theta)$ and $x(0^+)/x(0^-)=s(0^+)/s(0^-)$.

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